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THE

DOCTRINE

OF

FLUXIONS:

Not only Explaining

The ELEMENTS thereof,

But alſo its

APPLICATION and USE

In the ſeveral Parts of MATHEMATICS
and Natural PHILOSOPHY.

The Third Edition, corrected and greatly enlarged.

Ornari res ipſa negat contenta doceri.

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v. II

The PREFACE.

TO ſay any thing in Praiſe of the Method of Fluxions, or of its Dignity and Rank among the Mathematical Sciences, would be as needleſs as to deſcribe the Excellency of bright Sun-ſhine above the twinkling Light of the Stars; ſince any one who is acquainted with the Sciences will allow it to be a Method of Calculation incomparably ſuperior to all other Methods that ever were known or found out; and beyond which nothing further is to be hoped or expected. It lends its Aid and Aſſiſtance to all the other Mathematical Sciences, and that in their greateſt Wants and Diſtreſſes: It opens and diſcovers to us the Secrets and Receſſes of Nature, which have always before been locked up in Obſcurity and Darkneſs. To this all the noble and valuable Diſcoveries of the laſt and preſent Age are entirely owing: And by this Method Sir Iſaac Newton, *the worthy Inventor, determined and ſettled the Syſtem of the whole viſible World.*

The Uſe and Application of FLUXIONS *are exceedingly extenſive; for example, in Trigonometry, it teaches the Computation of Sines, Tangents and Secants; in Arithmetic, the Calculation of Logarithms; in Geometry, drawing Tangents to Curves, finding their Curvatures, their Lengths and Quadratures, the Surfaces and Solidities of Bodies; in Mechanics and Philoſophy, the Inveſtigation of the Centers of Gravity and Oſcillation, the Vibration of Pendulums, the Laws of Centripetal Forces, the Times, Velocities, and Spaces deſcribed by Bodies acted upon by any Forces, the Motions and Reſiſtances of Bodies in Mediums,* &c. *Theſe are ſome of the numberleſs Inſtances, wherein Fluxions are applied with ſuch wonderful Succeſs. And though ſome few of theſe may be (and actually have been) hammer'd out with great Labour and Difficulty by other Methods; yet the Proceſs of none of them can in the leaſt be compared with that Beauty, Simplicity, and charming Elegance, with which* *the Method of Fluxions performs all theſe Things. In ſhort, the Method of Fluxions is capable of reſolving ſuch Difficulties as raiſe the Wonder and Surpriſe of all Mankind, and which would in vain be attempted by any other Method whatſoever. So that it is juſtly eſteemed the greateſt Work of Genius, and the nobleſt Thought that ever entered the human Mind.*

The Method of Fluxions is founded upon this moſt ſimple and obvious Principle, viz. *that any Quantity may be ſuppoſed to be generated by continual Increaſe, after the ſame Manner that Space is deſcribed by local Motion. The great and noble Inventor tells us, that in this Method* he conſiders Things as generated by continual Increaſe, after the Manner of a Space which a Thing or Point in Motion deſcribes. *Now the Conception of this is exceeding eaſy and natural; for we every Day ſee with our own Eyes, all Kinds of Lines and Figures deſcribed by the Motion of Bodies: This Principle then will be eaſily admitted. And further, ſince we alſo ſee by Experience, that theſe very Lines and Figures are deſcribed, ſome with greater Degrees of Velocity, ſome with leſs, ſome with Motions continually accelerated or retarded, and ſome with uniform Motions: We ſhall eaſily underſtand, that any one of theſe Lines or Spaces has in every Point of its Deſcription a certain Degree of Increaſe determinate in itſelf, and peculiar to that Point, and which is the ſame with the Velocity of the Thing that deſcribes it. And to determine this Velocity, or this Degree of Increaſe, in any given Point of the generated Quantity, is the ſame Thing as finding the Fluxion of a propoſed variable Quantity, and is the Foundation of all the Arithmetic of Fluxions. And to determine this truly is of the greateſt Conſequence for eſtabliſhing the Theory.*

Many Diſputes and Objections have been advanced againſt the Truth of the Method of Fluxions; and amongſt theſe Diſputants, as it commonly happens, thoſe have been the moſt inveterate, who underſtood the leaſt of the Matter. To anſwer all the Cavils that have been offered will be of little Conſequence in any thing, and of none at all for *ſettling the true Notion of Fluxions. Therefore, inſtead of that, I ſhall, by following Nature as cloſely as I can, endeavour to give the unprejudiced Reader a clear and true Idea thereof; and then perhaps he will be able to judge for himſelf, whether the Principles of this noble Art be capable of Demonſtration or not. In order to this, let us aſſume what has been before laid down, that any generated, flowing Quantity is analogous to a Line deſcribed by a moving Point, and that the Velocity of this Point in any Place repreſents the Fluxion of that Quantity in the correſpondent Place of the Fluent; now I ſhall conſider the Generation of this Line, inſtead of the Fluent, as being more eaſily underſtood.*

It is the general Practice in Mechanics, to meaſure the Velocity of a Body by the Space uniformly deſcribed in a given Time. For Velocity being that *by which a Body is carried through a given Space in a given Time, therefore Velocity muſt be looked upon as the proper efficient Cauſe of the Space deſcribed; and the Space deſcribed the adequate Effect of that Cauſe. Now ſuppoſe a right Line deſcribed with any Sort of Velocity, accelerated, or retarded, at Pleaſure, and that we would enquire what is the Velocity of it in any given Place. If we take a ſmall Part of the Line, which the moving Point deſcribes juſt before it arrives at that Place, and call it an Increment, and ſuppoſe it to be deſcribed in a very ſmall given Time; then this Increment will nearly meaſure the Velocity of the deſcribing Point at the Place propoſed, and is ſufficient to give a vulgar Notion of the Degree of Velocity required. Now if this right Line was deſcribed uniformly, this would accurately meaſure the Velocity. But ſince that Increment is deſcribed with a Velocity, by Suppoſition, continually variable, therefore this Notion we have here obtained is to be corrected; the firſt Notions we get of any Subject are generally incorrect, and demand a nicer View, and a more accurate and philoſophical Examination, before we can acquire Notions that are perfect and adequate. Here then it will be very evident, if we take ſtill a leſſer and a leſſer Increment, by which the Velocity is* *is meaſured, as the Point ſtill draws nearer the propoſed Place; we approach nearer and nearer to a uniform Velocity, till the Difference be leſs than any aſſignable: And this Increment will differ from the true Meaſure of the Velocity, by leſs than any given Difference: And as this Increment continually diminiſhes, till at laſt it vaniſh, it approaches continually to that Meaſure, till the Difference vaniſhes with it.*

Now although by diminiſhing the Increment at Pleaſure we can approach within any Degree of Exactneſs to the Velocity required, yet ſince no Increment can be taken ſo ſmall, but it is ſtill further diviſible ad infinitum; *and ſince the Velocity is by Suppoſition continually variable, it is plain, there can be no two Points of this Increment in both which the Velocity is accurately the ſame. It is therefore moſt manifeſt, that the Velocity here enquired after is peculiar to one only indiviſible Point; and that Point is the Place where the Increment ends, or vaniſhes into nothing. Here then we ſee plainly, that the Velocity in any given Point of the Line deſcribed (or, which is the ſame Thing, that the Fluxion in any given Point of a generated Quantity) has a certain, fixed, determinate Value, proper to that Point of it alone: And this furniſhes the Mind with that accurate abſtract Idea, which we ought to form of this Velocity or Fluxion. And here we may obſerve, that this Degree of Velocity (or Fluxion) we have here been conſidering, and which continues but a Moment, differs from the ſame Degree of Velocity (or Fluxion) which continues for any given Time, and by which a given Space is actually deſcribed; theſe, I ſay, differ no otherwiſe than as a Cauſe in* POWER *differs from a Cauſe in* ACT.

Here a metaphyſical Diſputant may demand, how it comes to paſs, that any Velocity which continues *for no Time at all, can poſſibly deſcribe any Space at all; or whether its Effect be abſolutely nothing, or an infinitely ſmall Quantity, or what it is. Here then it is, that our Reaſon is at a Stand, and the human Faculties are quite confounded, loſt, and bewildered. We are puzzled* *and perplexed by endeavouring to examine into the Nature of* we do not know what, *nor whether it is ſomething or nothing: And at beſt is ſome ſuch ſubtile, fleeting Thing, as the Mind can lay no Hold on, nor form any Idea of. Now whether ſuch ſubtile Queſtions will be ever determined, or not, yet there is one Refuge for us,* viz. *that it is nothing at all to our Purpoſe what they are: And therefore we may ſafely leave theſe deep Speculations to thoſe that have more Buſineſs with them. The Method of Fluxions has no Dependence on theſe myſterious Diſquiſitions. What I apprehend the Method of Fluxions to be concerned in, is, not what any ſingle abſtract Velocity can deſcribe or generate of itſelf, but what a continual and ſucceſſively variable Velocity can produce in the whole. And here I think no Reaſon can be aſſigned, why a variable Cauſe ſhould not produce a variable Effect, as well as a permanent Cauſe a permanent and conſtant Effect. For ſince every Effect has a co-inſtantaneous Exiſtence with its efficient Cauſe, and is always perfectly connected with it; all the Difference can only be this, that the continual Variation of the Effect muſt always depend on, and be proportional to, the continual Variation of the Cauſe that produces it. And this will always be true, though we have no Ideas at all of the perpetually ariſing Increments, or their Magnitude in their naſcent or evaneſcent State, that have ſo much, and to ſo little Purpoſe, confounded and puzzled the mathematical World. And whether we can or we cannot conceive the formal Nature or Manner of exiſting of a Thing juſt ariſing out of nothing, or beginning to be; or whether a naſcent or evaneſcent Quantity be any thing or nothing; yet the Truth of the Method of Fluxions will ſtand juſt as it did. But theſe ſort of Diſputes have been artfully introduced for no other Purpoſe but to involve the Subject in Obſcurity, to darken the Reader's Judgment, and thereby to miſlead and divert him from purſuing the principal Buſineſs in Hand, that is, from conſidering the proper Evidence on which alone this Doctrine is founded; by inſinuating, that the Knowledge* *of theſe Things is eſſential to the very Nature and Foundation of Fluxions: When, it is evident, all that the Method of Fluxions does or ever did propoſe, being either to determine the Velocity (or Fluxion) wherewith a generated Quantity increaſes in any given Point; or elſe to ſum up all that has been deſcribed or generated by ſuch continually variable Fluxion, during any given Time, or to any given Point of the Fluent or generated Quantity. Theſe two Things alone are the two Baſes on which this noble Structure (the Method of Fluxions) is to be erected. All metaphyſical Speculations, of what Nature ſoever, having no Buſineſs here.*

I ſhall now bring an Inſtance or two out of the Phænomena of Nature, which will help the Reader's Notions a little; and will ſhew, that what has been ſaid before, concerning the Nature and Idea of Fluxions, is really true, and agreeable to the Nature and Conſtitution of Things. Let a heavy Body deſcend through a perpendicular Height of $16\frac{1}{12}$ *Feet in one Second of Time, according to the* Gallilean *Hypotheſis of Gravity; then at the End of this Second of Time, the Body has acquired a Velocity of* $32\frac{1}{6}$ *Feet in a Second; which therefore is accurately known. Now take any Point A in the right Line, at any given Diſtance from the Place the Body fell from, and the Velocity which the falling Body has in the Point A may be moſt accurately computed. But take any Point above A, though at ever ſo ſmall a Diſtance, if it be diſtant at all from A, and the Velocity in that Point will always be ſomething leſs than in the Point A. And in like manner the Velocity at any Point below A, though indefinitely near it, will be ſomething greater than in A; and therefore it is plain, that to the Point A, there belongs a certain determined Degree of Velocity, which belongs to no other Point in the whole Line, and this is accurately the Fluxion of that right Line in the Point A; and is the Velocity with which the Body would proceed, if the Force of Gravity ſhould be ſuppoſed immediately to ceaſe when the Body arrives at A, and to act no longer.*

Let there be a glaſs Tube, open at both Ends, and whoſe Concavity is of different Diameters in different Places, let it be immerſed in a running Stream of clear Water, ſo that the Water may flow freely through it, and always fill the Tube. Then it is evident, that in different Places of the Tube, the Velocity of the Water will be reciprocally as the Squares of the Diameters of the Tube, in theſe Places, and will therefore be different. Therefore if you mark any Place in the Side of the Tube, and ſuppoſe a Plane to paſs through the Tube perpendicular to the Axis, or to the Motion of the Water, then the Water will always paſs through this Section with a certain determinate Velocity. But ſuppoſe another Section to be drawn, though ever ſo near the former, then (by reaſon of the ſuppoſed different Diameters) the Water flows through this with a Velocity different from that it did at the former: And therefore that given determinate Velocity belongs only to one ſingle, indiviſible Point, or Section of the Tube, and this is the Fluxion of the Space which the Fluid deſcribes at that Section; and is that uniform Velocity with which the Fluid would *continue to move, if the Diameter continued the ſame through the ſucceeding Part of the Tube. Something like this may be obſerved in a River, for there the Velocity is greateſt, where the Dimenſions are leaſt, and leſs where theſe are greater.*

Again, let a hollow Cylinder be filled with Water, and let it flow freely out through a Hole at the Bottom of it. It is well known, that the Velocity of the effluent Water depends on the Height of the Water within the Cylinder; and therefore, ſince the Surface of the incumbent Water continually deſcends without any the leaſt Stop, the Velocity of the effluent Stream will continually decreaſe, till it all be run out. Therefore it is plain, there can be no two Moments of Time, ſucceeding each other ever ſo nearly, wherein the Velocity of the running Water is preciſely the ſame. And therefore the Velocity that the effluent Water has at any given Point of Time, belongs only to that one particular, indiviſible Moment of Time, *and no other: And this is accurately the Fluxion of the Fluid flowing out at that Moment of Time. Now if preciſely at that Moment you begin and continue to pour more Water into the Cylinder, ſo that the Surface of the Water may deſcend no lower, but keep its Place; then the effluent Water will alſo retain its Velocity, and continue to be the Fluxion of the Fluid as before. Now theſe are the genuine Effects and Operations of Nature itſelf; and do, in a manner viſibly, confirm the Truth of what has been ſaid of the Nature of* FLUXION.

From theſe Examples, and many more that might be produced, it is clear to me, that it is an eſſential Property of the Fluxion of a generated Quantity, that it does not retain any one determined Value for the leaſt Space of Time whatever; but at the Moment it arrives at that Value, the ſame Moment it leaves it again; ſo that it only paſſes gradually and ſucceſſively through all the indefinite Degrees contained between the two extreme Values which are the Limits thereof, during the Generation of the Fluent: That is, in caſe the Fluxion be variable at all; but if it is invariable, the extreme Values, and all the intermediate Degrees are but one and the ſame Value. And therefore, although any determinate Degree of Fluxion does not continue at all; yet every Fluent has (intrinſically) in itſelf, ſome determinate Degree of Fluxion, at every determinate, indiviſible Moment of Time.

It being now, I ſuppoſe, made evident, that every generated Quantity has every where a certain Rate *of increaſing (called its* Fluxion) *whoſe abſtract Value is determinate in itſelf, at any determinate Point of that Quantity: Therefore to find out its Value, or its Ratio to any other Fluxion, is a Problem ſtrictly geometrical. It remains to enquire in general how we muſt compute this Value. And here the only, or at leaſt the moſt natural Way is, to get the Proportion of the Increments generated by the Fluxions in all Suppoſitions of Magnitude of theſe Increments, and from thence collect the Ratio they firſt begin with. When the Fluxions and Moments*

Moments of the ſimple literal Quantities are deſigned by proper Symbols; it will be eaſy by the binomial Theorem to find the contemporary Increment of any compound Quantity; for that will be expreſſed in the Form of a Series. Now although it is evident, that this Increment of the compound Quantity is not accurately the Fluxion of it in any determinate Point, becauſe that very Increment is generated by a Fluxion continually variable; yet it is as evident *that it continually approaches to it, by continually diminiſhing the Increments of the ſimple Quantities. Here then will be had in general, the Ratio of the Fluxion of a ſimple Quantity to the Fluxion of that compound Quantity, and in the loweſt Terms, and that as near the Truth as we pleaſe, whilſt we ſuppoſe ſome, though very ſmall, Increment actually deſcribed. But ſince the Ratio of theſe Fluxions is required for, and belongs only to, ſome one indiviſible Point of the Fluent, that is, in the very Beginning of the Increment, or when there is no Increment at all generated; therefore in this particular Caſe making the Values of the ſimple Increments nothing, which before was expreſſed in general, and then all the Terms wherein they are found will vaniſh, and what is left will accurately ſhow the Relation of the Fluxions, for that ſingle indiviſible Point where the Increment is ſuppoſed firſt to commence, or was required. For this abſtract Value of the Fluxion belongs to no more Points than one of the Fluent; and therefore of Conſequence the Moments muſt be made to vaniſh, after we have ſeen by the continual Diminution thereof, whither the Ratio tends, and what it continually converges to; which will be as viſible to every Body as the very Characters it is written in. And if any one ſhould doubt of the Truth of this, I ſhould for ever deſpair of convincing him of any Thing at all. The Increments here muſt neceſſarily be made uſe of, not to determine* their *Magnitude, as ſome have abſurdly imagined, but as a* Medium *in our Reaſoning, to diſcover the Quantity of the Cauſe that produces them, they being the continual Effects of the Fluxions; and how can we* *judge of the Force or Efficacy of a Cauſe, without conſidering the Effect that it does or could produce: For that like Cauſes are proportional to their Effects can never be denied, except by thoſe that can deny any Thing. This Way of Reaſoning and Method of Demonſtration then muſt be exceedingly clear and convincing to all that are duly qualified to examine and conſider it, and do not with a moſt unaccountable Obſtinacy, and invincible Prejudice, reſolve to yield to no Reaſon at all, though laid before them as clear as the Sun.*

And here it may be worth obſerving, that in the Proceſs of this Demonſtration, the Terms which vaniſhed out of the Increment of the compound Quantity, did plainly ariſe from, and was generated by the Variation *of the Fluxion of that compound Quantity; and the remaining Quantities alone are thoſe generated by the Fluxion itſelf.*

The Eaſineſs and Simplicity of this Method of Demonſtration is no ſmall Argument for its Truth and Perfection. The Simplicity of Truth is its great Beauty. And by this Mark it here proves itſelf to be the genuine Off-ſpring of Nature and Truth. But if any Perſons will not aſſent to the Truth of theſe Principles, I would have them ſuſpend their Judgments, leſt they make it appear that they have no Judgment at all. In the mean Time let them compare the Reſults and Concluſions obtained by the Method of Fluxions, with the like Concluſions obtained by other undiſputed Principles or Methods of Calculation; and if theſe Reſults continually agree, then it is a convincing Proof (at leaſt à poſteriori) *that the Principles from whence they are deduced muſt be equally true. But if any Perſon that plainly diſcovers himſelf unacquainted with mathematical Principles, ſhall, out of his Averſion to theſe Sciences, cavil and diſpute againſt the Principles here laid down, and which he underſtands nothing of; and endeavour to put the Iſſue on ſuch a footing, as neither himſelf nor any Body elſe can underſtand any Thing about it, by running the Account of it into the dark: I think it can be of no Conſequenc* *ſequence at all to trouble a Man's ſelf with this Sort of* Anti-Mathematicians. *Ignorance, Malice, or Prejudice can deſerve no Notice.*

And thus much I choos'd to ſay here by Way of Preface, rather than in the Book itſelf, (which I would not encumber with needleſs Diſputes) concerning the Nature and Principles of Fluxions; a Thing eaſy enough to be underſtood, and rendered difficult, more by the intricate Diſputes that have been dragged into it by the Enemies of Science, than from the Nature of the Thing itſelf. The divine Newton *(whoſe Works will laſt as long as the Sun and Moon) clearly ſaw that this Matter did not at all require to be built upon any metaphyſical Speculations; he, by expreſſing the ſimple Moments by* general *Characters, did thence derive, by infinite Series, the Moments of compound Quantities; from whence he gets the Proportion of the Fluxions for any indetermined Values of theſe Moments, from which* general Proportion *he at laſt gains their Proportion for that* particular *Caſe where the Moments firſt begin, or at laſt vaniſh into nothing. And thus he has given a Demonſtration extremely eaſy and compleat in it ſelf.*

If Arts and Sciences of many hundred Years ſtanding receive daily Improvements and Additions, it cannot be ſuppoſed that this moſt ſublime Art of all, found out but Yeſterday, can be arrived at Perfection all on a ſudden. If this Art be ſo exceedingly uſeful and valuable, it certainly deſerves the Pains and Attention of the learned Mathematicians. And the World muſt expect, that the Beauty of this Method will excite them to lend all their Aſſiſtance towards the Advancement of ſo noble a Branch of Learning, whether it be in improving the Theory, or facilitating the Practice. Therefore I hope I ſhall be excuſed at leaſt, if, among others, I endeavour to contribute a little towards this great End.

The following Book is divided into three Sections: In the firſt *are laid down ſuch univerſal Propoſitions, as are the Foundation of all that Doctrine. The* ſecond *Section applies theſe Principles to the Solution of the moſt general*

ral Problems, or thoſe of moſt frequent Uſe in the Mathematicks: And here many of the Problems ſo often done by others, are reſolved by Methods entirely new, and, I think, more ſimple; and therefore will by many Perſons be more eaſily apprehended in this Form. And becauſe the Reſolution of phyſical Problems has been little touched on by others, I have added the third *Section, where you have the Inveſtigation of ſome of the chief phyſical Problems in the Phænomena of Nature. And in this Section, it may perhaps pleaſe a Reader of the* Principia (*the greateſt human Production that ever appeared in the World*) *to ſee many of the Author's ſubtle Problems reſolved by his own Analyſis.*

It is not in the leaſt pretended that all Things here treated of are new; for I have collected many Things which I thought material for forming this Doctrine *into a regular Syſtem; and what was wanting I endeavoured to ſupply as well as I could; not that I take it to be perfect: For there are many* Deſiderata *ſtill wanting to compleat the Science. Becauſe the Method of finding Fluents by the Tables is exceeding compendious and uſeful, and has yet been but very ſlightly paſſed over by the Writers on this Subject: I have been at the Pains (which was not a little) to compoſe a new Table, whoſe Uſe will appear upon trial to be far more eaſy and intelligible than any extant; and no leſs extenſive. And for the Explanation and Uſe of it, I have given a vaſt Variety of Examples throughout the whole Book. Yet I have not omitted the moſt general known Rules for finding the Fluents by infinite Series; and have inſerted the general Forms of them in the Table. In the Uſe of which Table there is not the leaſt Difficulty, there being nothing required but a bare Subſtitution of Quantities. But as to the Reſolution of Problems by infinite Series, I have been more ſparing of that, becauſe it has been well proſecuted by others. I am not ignorant, that (by the Method of Tranſmutation of Fluxions) this Table might have been further extended, and other more compounded Forms might have been inſerted. But, conſidering how ſeldom*

ſeldom theſe come in Uſe, I thought it needleſs to carry it any further. I have all along, in my Calculations, uſed diſtinct Characters for the Fluxions and Moments, ſince they ought not to be confounded together. And in moſt Problems (except ſuch where the Reaſoning is ſo obvious as not to need it) I have uſed both of them; beginning firſt with the Moments, and ſubſtituting at laſt the Fluxions for the evaneſcent Moments which are proportional thereto.

In all theſe Things I have deſignedly been very ſhort. For the general Rules and Methods of Operation being laid down in as few Words as poſſible, the Examples will explain their Meaning and Uſe. In the mathematical Sciences I have taken general Methods to be beſt; and they that deal in the detail of Things, and ſpin them out to an unneceſſary Length, making thereby a pompous ſhow of Words only, do certainly miſpend the Time of their Readers: Since one great End to be aimed at in the Sciences, is to abridge and reduce them to the moſt general and conciſe Rules.

As I am not conſcious of any Faults I have committed in this Treatiſe, ſo I hope they are but few. But in ſuch a vaſt Variety of Things of the moſt intricate Nature, it is hardly poſſible but ſome will eſcape. Therefore I muſt beg of the courteous and good natur'd Reader (for whom alone it was written) that he will rather kindly inform me of my Errors and Defects, than cenſure and condemn the Book. For as Truth is what I ſeek, I ſhall with Pleaſure retract or correct any Thing I have written, when it appears inconſiſtent with that or the Reaſon of Things.

Laſtly, let me acquaint the Reader, that it is indiſpenſably required, that he perfectly underſtand Arithmetic, Geometry, and Algebra in all their Parts and Improvements, the Methods of Series, Doctrine of Proportions, Nature of Logarithms, Mechanics, and Laws of Motion, &c. *all which are to be learned from theſe particular Sciences to which they belong. For I am clearly perſuaded, that it is the beſt Method to treat every Science* *diſtinct and entire by itſelf, without the Mixture or Interpoſition of any Thing foreign to the Subject. And therefore in this Treatiſe I have delivered nothing but the pure* Doctrine of Fluxions *alone. It would be but loſt Labour for any Perſon unacquainted with theſe* Precognita, *to ſpend any Time in reading this Book; or indeed to attempt to read any ſuch like Treatiſe with any tolerable Judgment. The Conſequence would be, that either the Author or the Difficulty of the Subject muſt be blamed, as is always the Caſe; but never the Reader. But then if he comes thus prepared, this will make every Thing eaſy and pleaſant, and he will then find few Difficulties here, but what he will eaſily ſurmount. All which I ſubmit to the Peruſal of the candid and judicious Reader.*

W. Emerſon.

P. S. *In this ſecond Edition I have made many Alterations and Additions in ſeveral Parts of the Book, which are too tedious to mention: But the Additions of principal Note are theſe, in Sect. I. ſeveral more Examples of transforming Fluxions are added to Prop.* 9. *as alſo more Examples of finding Fluents, to Prop.* 10. *There is alſo ſome farther Explication of the Uſe of the Table; likewiſe Ex.* 10 *and* 11 *to Prop.* 13, *are enlarged, and more Examples added. In Sect. II. Prob.* 1. *is enlarged, amd two more Problems added at the End. Likewiſe I have added a new Problem at the Beginning of Sect. III. and ſeveral more phyſical Problems at the End of it, with two new Copper Plates.*

W. E.

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THE DOCTRINE OF FLUXIONS.

POSTULATUM.

That any Quantity may be ſuppoſed to be generated by continual Increaſe.

HERE Quantities are conſider'd, not as compoſed of an infinite Number of conſtituent Parts, but as deſcribed by a continued Motion. Thus a Line is deſcribed by the Motion of a Point; and a Rectangle may be conceiv'd to be generated by the Motion of one Side along the other; and Time proceeds by a regular Flux. And all other Quantities may (by Analogy) be conceived to be generated after the ſame Manner.

DEFINITIONS.

Definition I.

Quantities generated by a continual Increaſe are called *Fluents* or *Flowing Quantities*. Thoſe Quantities that always retain the ſame Value are called *given*, *conſtant*, *ſtanding*, or *invariable Quantities*; and thoſe that are continually changing their Value are called *variable* or *indetermin'd Quantities.*

Thus in a Circle the Diameter is a conſtant Quantity; and the verſed Sine and correſpondent Sine are variable Quantities: Alſo in a Parabola, the Latus Rectum is a conſtant Quantity, and the Abſciſa and correſpondent Ordinate are variable Quantities.

Def. II.

The Velocity of the Increaſe of any generated Quantity, or the Degree of Quickneſs (or Slowneſs) wherewith the new Parts of it continually ariſe, is called its *Fluxion.*

Thus when a Line is generated by the Motion of a Point, the Line itſelf is the Fluent, and the Velocity of the moving Point is ſtrictly its Fluxion. But Velocity is never properly aſcribed to any Thing but local Motion, and is uſed in this Definition, rather to *deſcribe* what is meant by the Word Fluxion, than to *define* it. Velocity is the ſame in a *particular* Senſe in Relation to the Space deſcribed, as Fluxion is in a *general* Senſe in Relation to the Fluent generated thereby. Velocity is allow'd by all to be a ſimple Idea, and ſo is Fluxion too. When a Man conſiders the Generation of ſeveral Quantities, after this Manner, he will find ſome to increaſe faſter, others ſlower; and conſequently that there are comparative Velocities (or Fluxions) of Increaſe during their Generation: And thus he will by Degrees get the Idea of a Fluxion; but without ſuch attentive Conſideration, he will never be the wiſer for all the Words in the World.

Def. III.

The indefinitely ſmall Portions of the Fluent which are generated in any indefinitely ſmall Portions of Time are called *Moments* or *Increments.* Or if the Fluent decreaſes, the Portions continually deſtroy'd are called *Decrements.*

Theſe Moments are the immediate Effects of the Fluxions, and are thoſe Quantities by the continual Acceſſion of which the Fluent increaſes and grows bigger and bigger: That is, any Moment conſider'd alone is the adequate Effect of ſome ſingle determinate Fluxion which is (conſider'd as) its generating Cauſe. Therefore the Moments and Fluxions ought not to be confounded together, ſince the Moments (being generated by the Fluxions) are as different from the Fluxions, as any Effect is different from its Cauſe.

Def. IV.

The *Velocity*, *Variation*, or *Quickneſs* of *Increaſe* (or *Decreaſe*) of any Fluxion is called the *ſecond Fluxion*; likewiſe the *Variation* or Quickneſs of *Increaſe* of the ſecond Fluxion is called the *third Fluxion*, &c.

As in the Generation of any Fluent, the different Parts of it may be generated faſter or ſlower, that is, its Fluxion at different Times may be unequal; ſo there muſt be Degrees of Variation by which it is continually changing, that is, it muſt have a ſecond Fluxion. And in like Manner this ſecond Fluxion may alſo be continually variable, and therefore muſt have a certain Degree of Variation in every Point, or a third Fluxion. And ſo on.

Def. V.

Contemporary Fluents are thoſe which are ſuppoſed to be generated together or in equal Times; or which begin together and end together.

Def. VI.

In any Fluxionary Equation, a *Quantity* of the *firſt Order* is that which has only one firſt Fluxion in it; a *Quantity* of the *ſecond Order* has either one ſecond Fluxion or two firſt Fluxions: *Quantities* of the *third Order*, are third Fluxions, product of three firſt Fluxions, product of a firſt and ſecond Fluxion, *&c.*

NOTATION.

1. The firſt Letters of the Alphabet, a, b, c, *&c.* are generally put for ſtanding Quantities; and the laſt x, y, z, *&c.* for variable or flowing Quantities.

2. If x or y be put for any fluent or variable Quantity, then the ſame Letter with a Point over it $\dot{x}$, $\dot{y}$, denotes the Fluxion of x or y reſpectively; and the ſame Letters twice pointed $\ddot{x}$ and $\ddot{y}$, are the Fluxions of $\dot{x}$ and $\dot{y}$, or the ſecond Fluxions of x and y: thus $\dddot{x}$ and $\dddot{y}$ are the third Fluxions of x and y: Likewiſe $\ddddot{x}$ and $\ddddot{y}$ denote the fourth Fluxions of x and y, *&c.* alſo the Fluxion of a or b is 0.

3. Again, $\overset{\prime}{x}$ denotes the Moment or Increment of x, and $\overset{\prime}{y}$ the Moment or Increment of y; likewiſe $\overset{\prime\prime}{x}$ and $\overset{\prime\prime}{y}$ denote the Moments of the Moments, or the ſecond Moments of x and y, *&c.*

4. To the common Algebraic Characters already receiv'd I add this $\propto$, which ſignifies a general Proportion; thus, $A \propto \frac{BC}{D}$, ſignifies that A is in a conſtant Ratio to $\frac{BC}{D}$; that is (if a, b, c, d be other Values of theſe Quantities) $A : \frac{BC}{D} :: a : \frac{bc}{d}$; and thus every general Proportion is to be underſtood.

AXIOM.

Quantities, which in any finite Time continually converge to Equality, and *before* the End of that Time, approach *nearer* to one another than by any given Difference, do at laſt become equal.

If any ſhould think this not clear enough to paſs for an Axiom, he may conſider it thus; let D be their ultimate Difference, therefore they cannot approach nearer to Equality, than by that given Difference D, contrary to the Hypotheſis; which Suppoſition is abſurd in all Caſes except when D is nothing.

SECT.

SECT. I.

The fundamental Principles and Operations of FLUXIONS.

PROP. I.

The Fluxion of any Fluent or generated Quantity is equal to the Sum of the Fluxions of all the Roots or Sides, each multiply'd continually by the Index of its Power, and by the given Fluent divided by the ſaid Root or Side.

DEMONSTRATION.

1. LET the Fluent be $bx^m y^n$; now its Fluxion muſt be a Thing real and determinate in itſelf, otherwiſe we are ſeeking that which has no Exiſtence. By the Notation, $\dot{x}$ and $\dot{y}$ are the Fluxions of x and y; and will produce Effects, that is, will generate Moments proportional to themſelves whilſt they retain their Values, which therefore may be expreſſed by $o\dot{x}$, $o\dot{y}$.

2. Now by the Poſtulatum, theſe Moments will increaſe the Quantities x, y, which therefore will become $x+o\dot{x}$, and $y+o\dot{y}$.

3. Therefore the Fluent $bx^m y^n$ will now become $b \times \overline{x+o\dot{x}}^m \times \overline{y+o\dot{y}}^n = b \times : x^m + mx^{m-1} o\dot{x} + px^{m-2} o^2 \dot{x}^2$, &c. $\times : y^n + ny^{n-1} o\dot{y} + qy^{n-2} o^2 \dot{y}^2$, &c. where p, q are given Quantities.

4. The laſt Quantity being actually multiply'd, and $bx^{m}y^{n}$ ſubſtracted from it; we ſhall have $bmx^{m-1}y^{n}o\dot{x} + bnx^{m}y^{n-1}o\dot{y} + bpx^{m-2}y^{n}o^{2}\dot{x}^{2} + bqx^{m}y^{n-2}o^{2}\dot{y}^{2} + bmnx^{m-1}y^{n-1}o^{2}\dot{y}\dot{x} +$, &c. for the Moment of $bx^{m}y^{n}$.

5. Therefore the Moment laſt found being divided by the indefinite Quantity o, will give the Fluxion of $bx^{m}y^{n}$ equal to $bmx^{m-1}y^{n}\dot{x} + bnx^{m}y^{n-1}\dot{y} + bpx^{m-2}y^{n}o\dot{x}^{2} + bqx^{m}y^{n-2}o\dot{y}^{2} + bmnx^{m-1}y^{n-1}o\dot{y}\dot{x}$ $+$, &c. and this is the Fluxion (or Velocity) wherewith the foregoing Moment is (or may be) uniformly generated.

6. But ſince the (Velocity or) Fluxion is required wherewith that Moment firſt ariſes, in this Caſe the Moments $o\dot{x}$ and $o\dot{y}$ will alſo be juſt ariſing and therefore nothing, and conſequently o will be nothing, and therefore all the Terms wherein it is found will be nothing:

7. Therefore the Fluxion of $bx^{m}y^{n}$ at that Moment of Time is accurately $bmx^{m-1}y^{n}\dot{x} + bnx^{m}y^{n-1}\dot{y}$. Q. E. D.

Otherwiſe thus.

1. Let $\acute{x}$ and $\acute{y}$ be very ſmall Increments uniformly generated by $\dot{x}$ and $\dot{y}$, in a very ſmall Time.

2. The Quantities x, y, $bx^{m}y^{n}$ then at the End of that Time are become $x+\acute{x}$, $y+\acute{y}$, and $b \times \overline{x+\acute{x}}^{m} \times \overline{y+\acute{y}}^{n}$; and the Increments generated in that

Time are $\acute{x}$, $\acute{y}$ and $bmx^{m-1}y^{n}\acute{x} + bnx^{m}y^{n-1}\acute{y} + bpx^{m-2}y^{n}\acute{x}^{2} + bqx^{m}y^{n-2}\acute{y}^{2} + bmnx^{m-1}y^{n-1}\acute{x}\acute{y} +$, *&c.* $=$ M by Subſtitution, where note p, q are given Quantities.

3. Now ſince $\dot{x} : \dot{y} :: \acute{x} : \acute{y} = \frac{\dot{y}\acute{x}}{\dot{x}}$, becauſe the Effects of like Cauſes are proportional; therefore expunge $\acute{y}$ out of the Value of M, and we ſhall have the Increment of x to the Increment of $bx^{m}y^{n}$ as $\acute{x}$ to M or as 1 to $\frac{\text{M}}{\acute{x}}$; that is, as $\dot{x}$ to $bmx^{m-1}y^{n}\dot{x} + bnx^{m}y^{n-1}\dot{y} + bpx^{m-2}y^{n}\acute{x}\dot{x} + bmnx^{m-1}y^{n-1}\acute{x}\dot{y} + bqx^{m}y^{n-2}\times\frac{\acute{x}\dot{y}^{2}}{\dot{x}} +$, *&c.* $= \text{P} + \text{Q}\acute{x}$, *&c.* by Subſtitution, (putting P for the two firſt Terms, and $\text{Q}\acute{x}$, *&c.* for the reſt).

4. But $\dot{x}$ is the Fluxion of x, therefore $\text{P}+\text{Q}\acute{x}$ is the Fluxion of $bx^{m}y^{n}$. But ſince this laſt Fluxion $\text{P}+\text{Q}\acute{x}$ is not that with which the Increment begins to be generated, which we ſeek: It is evident that by diminiſhing the Time, or which is the ſame thing, diminiſhing $\acute{x}$, the Fluxion $\text{P}+\text{Q}\acute{x}$ continually converges to that Fluxion (or Velocity) wherewith the Increment firſt ariſes, and before $\acute{x}$ be diminiſh'd to nothing, is nearer to it than by any given Difference; and therefore by the Axiom when $\acute{x}$ is nothing, and conſequently $\text{Q}\acute{x}$ nothing, then P or $bmx^{m-1}y^{n}\dot{x} + bnx^{m}y^{n-1}\dot{y}$ will be the Fluxion of $bx^{m}y^{n}$. In like Manner it may be demonſtrated for any other Quantity. Q. E. D.

COR. 1. A Fluent can have but one Fluxion: Thus the Fluxion of x can only be $\dot{x}$; of ax^2, $2ax\dot{x}$, *&c.*

COR. 2. When the Fluxion of any Quantity is the ſame with a propoſed Fluxion, then that Quantity is its Fluent.

SCHOLIUM.

As clear and evident as this Propoſition is, yet it has been cenſur'd as falſe and erroneous; though the Perſons that object againſt its Truth were never able to tell us what the Error is, nor whether the Fluxion of any Quantity is greater or leſſer than is aſſign'd by this Propoſition.

For a further Illuſtration, let us ſuppoſe that x and y are uniformly diminiſhed by the very ſmall Increments $\acute{x}$ and $\acute{y}$. Then be reaſoning as before, we ſhall get the Fluxion (or Velocity) of the contemporary Moment of $bx^m y^n = bmx^{m-1} y^n \dot{x} + bnx^m y^{n-1} \dot{y} - bpx^{m-2} y^n \acute{x}\dot{x} - bmnx^{m-1} y^{n-1} \acute{x}\dot{y} - \frac{bqx^m y^{n-2} \acute{x}\dot{y}^2}{\dot{x}}$ *&c.* $= P - Q\acute{x}$, *&c.* Now by continually diminiſhing $\acute{x}$, it is manifeſt that $P - Q\acute{x}$, *&c.* will differ from the Fluxion the Moment firſt ariſes with, by a Quanty leſs than any aſſignable, before $\acute{x}$ be reduced to nothing.

Now when we ſee that whilſt $x + \acute{x}$ converges to x, that at the ſame Time $P + Q\acute{x}$, *&c.* converges to that Fluxion the Moment ariſes with, and differs in *Exceſs* from it by leſs than any given Difference, and ſtill differs the leſs, the leſs $\acute{x}$ is taken; and when $\acute{x}$ is infinitely ſmall, that Difference is infinitely ſmall in *Exceſs:* And ſeeing alſo that whilſt $x - \acute{x}$ converges towards x, likewiſe $P-Q\overset{\prime}{x}$, *&c.* approaches to the Fluxion the Moment vaniſhes with (or the ſucceeding Moment begins with, being the ſame the other converg'd to) and differs in *Defect* from it, by leſs than any given Deference, and ſtill differs the leſs as $\overset{\prime}{x}$ is leſs, and when $\overset{\prime}{x}$ is infinitely diminiſh'd, that Difference is infinitely ſmall in *Defect.* If any one after all this ſhould contend, that when $\overset{\prime}{x}$ is quite vaniſh'd and become nothing, that the Fluxion of bx^my^n is not accurately P or $bmx^{m-1}y^n\dot{x}+bnx^my^{n-1}\dot{y}$; I think no man ought to give himſelf any Concern at all for ſuch an Adverſary, or take any further Pains for his Conviction.

PROP. II.

If two Fluents or variable Quantities be equal to each other, or in a given Ratio; their Fluxions will be equal, or in the ſame given Ratio.

And if two fluxionary Quantities are equal or in a given Ratio; their contemporary Fluents will be equal or in the ſame given Ratio.

For ſince the Quantities always continue equal, or in a given Ratio; the continually ariſing Increments, and therefore the Fluxions proportional thereto, will neceſſarily be equal or in the *ſame* given Ratio. And *vice verſâ*, the Fluxions or generating Cauſes being always equal or in a given Ratio; their Effects or contemporary Fluents will therefore be equal, or in the ſame given Ratio. Q. E. D.

COR. 1. Two fluxionary Quantities may be equal, and their ſimple Fluents unequal. For (by this Prop.) only the contemporary Fluents can be equal. And therefore

COR. 2. A Fluxion may have an infinite Number of Fluents: thus the Fluent of $\dot{x}$ is x, $a+x$, $b+x$, $c+x$, $x-a$, $x-b$, $x-c$, &c.

COR. 3. If any Fluxion be equal (or nearly equal) to ſome other Fluxion, in ſome particular Caſe; then the one may be ſubſtituted for the other in any fluxionary Equation, and their Fluents will be equal in that particular Caſe, (or nearly ſo.)

COR. 4. If there be any Relation whatſoever between the Moments of ſeveral Quantities in their naſcent or evaneſcent State; the ſame Relation there is alſo between the Fluxions of theſe Quantities.

COR. 5. And therefore in any Equation between the Moments conſider'd as ariſing, the Fluxions may be ſubſtituted in their Room; and the contrary.

PROP. III.

Given an Equation containing the Relation of the flowing Quantities; to determine the Relation of their Fluxions.

SOLUTION.

1. If any Term contains only one variable Quantity; multiply its Index, the Fluxion of the Root, its Power whoſe Index is leſſened by 1, and the Coefficient of the Term continually, for the Fluxion of that Term:

2. If any Term involves ſeveral flowing Quantities, obſerve what Power of any variable Quantity is contain'd

tain'd in that Term, and multiply the Fluxion of that Power (found by Art. 1.) by all the other Quantities in the Term; do the ſame for all the variable Quantities; the Sum of all theſe Products is the Fluxion of that Term.

3. Repeat the ſame Operation for all the Terms in the Equation; and the Sum of all gives an Equation containing the Fluxions required.

4. In exponential Equations, or thoſe whoſe Exponents are variable; let X, Y, Z, *&c.* be the hyperbolic Logarithms of x, y, z, &c. then multiply the Index of any Quantity into the Logarithm of that quantity, and you'll have a logarithmic Equation, whoſe Fluxion is to be found by the foregoing Rules: Then expunge the logarithmic Fluxions $\dot{X}$, $\dot{Y}$, or $\dot{Z}$, by ſubſtituting their equals $\frac{\dot{x}}{x}$, $\frac{\dot{y}}{y}$, or $\frac{\dot{z}}{z}$; the Reaſon of which will appear hereafter by Prob. II. Sect. II.

5. Sometimes it may be convenient to divide the Equation given, by ſome of the indetermin'd Quantities contained in moſt of the Terms; or to ſubſtitute ſingle Letters for compound Quantities.

EXAMPLE 1.

Let $y=a+x-z-b-dx$; then the Fluxion is $\dot{y}=\dot{x}-\dot{z}-d\dot{x}$.

Ex. 2.

Let $x=ay^3$, then the Fluxion of this Equation is $\dot{x}=3ay^2\dot{y}$.

Ex. 3.

Let $y=ax^2$, the Fluxion is $\dot{y}=2ax\dot{x}$. Or in particular Numbers, let $a=10$, $\dot{x}=1$, then $\dot{y}=20x$. Now if $x=1$, then $\dot{y}=20$. If $x=2$, 3, 4, *&c.* then $\dot{y}=40$, 60, 80, *&c.*

Ex. 4.

Let $z=120bx^n$, the Fluxion is $\dot{z}=120nbx^{n-1}\dot{x}$.

Ex. 5.

Let $\sqrt{aa-xx}=y$, that is $\overline{aa-xx}|^{\frac{1}{2}}=y$; its Fluxion is $\frac{1}{2}\times\overline{-2x\dot{x}}\times\overline{aa-xx}|^{-\frac{1}{2}}=\dot{y}$, or $\frac{-x\dot{x}}{\sqrt{aa-xx}}=\dot{y}$. In Numbers thus, let $a=10$, $\dot{x}=1$, then $\dot{y}=\frac{-x}{\sqrt{100-xx}}$, and if $x=0$, then $\dot{y}=\frac{-0}{10}=0$. If $x=3$, $\dot{y}=\frac{-3}{\sqrt{91}}$. If $x=6$, then $\dot{y}=\frac{-3}{4}$. If $x=10$, $\dot{y}=\frac{-10}{0}=-$ Infinity; and ſo of others.

Ex. 6.

Let $v=bx^3y^2$, the Fluxion is $\dot{v}=3by^2x^2\dot{x}+2bx^3y\dot{y}$.

Ex. 7.

Let $\frac{x}{y}=v$, that is $x^1y^{-1}=v$; the Fluxion is $y^{-1}\dot{x}-1\times x^1y^{-2}\dot{y}=\dot{v}$, or $\frac{\dot{x}}{y}-\frac{x\dot{y}}{y^2}=\dot{v}$, that is $\frac{y\dot{x}-x\dot{y}}{yy}=\dot{v}$.

Ex. 8.

Let $x^3-ax^2+axy-y^3=0$. Its Fluxion will be $3x^2\dot{x}-2ax\dot{x}+ax\dot{y}+ay\dot{x}-3y^2\dot{y}=0$.

Ex. 9.

Let this Equation be given $y^2-a^2=x\sqrt{aa-xx}$; its Fluxion is $2y\dot{y}=\dot{x}\sqrt{aa-xx}-\frac{x^2\dot{x}}{\sqrt{aa-xx}}$, or $2y\dot{y}=\frac{aa\dot{x}-2x^2\dot{x}}{\sqrt{aa-xx}}$.

Ex.

Ex. 10.

Let $2y^3+x^2y-2cyz=z^3-3yz^2$ be given; then its Fluxion is $6y^2\dot{y}+2yx\dot{x}+x^2\dot{y}-2cy\dot{z}-2cz\dot{y}=3z^2\dot{z}-3z^2\dot{y}-6yz\dot{z}$.

Or thus, divide the given Equation by y, and you'll have $2y^2+x^2-2cz=\frac{z^3}{y}-3z^2$, whose Fluxion is $4y\dot{y}+2x\dot{x}-2c\dot{z}=\frac{3z^2\dot{z}}{y}-\frac{z^3\dot{y}}{y^2}-6z\dot{z}$; or $4y^3\dot{y}+2yyx\dot{x}-2cy^2\dot{z}=3yz^2\dot{z}-z^3\dot{y}-6y^2z\dot{z}$.

Ex. 11.

$x^3-ay^2+\frac{by^3}{a+y}-xx\sqrt{ay+xx}=0$, the Fluxion is $3x^2\dot{x}-2a\dot{y}+\frac{3by^2\dot{y}}{a+y}-\frac{by^3\dot{y}}{\overline{a+y}|^2}-2x\dot{x}\sqrt{ay+xx}$ $\frac{-ax^2\dot{y}-2x^3\dot{x}}{2\sqrt{ay+xx}}=0$. Or $3x^2\dot{x}-2ay\dot{y}+\frac{3aby^2\dot{y}+2by^3\dot{y}}{\overline{a+y}|^2}-\frac{4ayx\dot{x}+6x^3\dot{x}+ax^2\dot{y}}{2\sqrt{ay+xx}}=0$.

Ex. 12.

Let $\sqrt{ax+\sqrt{aa-xx}}=v$, then will $ax+\sqrt{aa-xx}=vv$, whose Fluxion is $a\dot{x}-\frac{x\dot{x}}{\sqrt{aa-xx}}=2v\dot{v}$, whence $\dot{v}=\frac{a\dot{x}}{2v}-\frac{x\dot{x}}{2v\sqrt{aa-xx}}$, or $\dot{v}=$

$$\frac{a\dot{x}-\frac{x\dot{x}}{\sqrt{aa-xx}}}{2\sqrt{ax+\sqrt{aa-xx}}}.$$

Ex. 13.

Let $z\dot{y}-a\dot{x}=0$; its Fluxion will be $\dot{z}\dot{y}+z\ddot{y}-a\ddot{x}=0$. Or thus, let $s=\dot{y}$, $t=\dot{x}$, then $\dot{s}=\ddot{y}$, and $\dot{t}=\ddot{x}$; and the Equation $z\dot{y}-a\dot{x}=zs-at=0$.

whoſe Fluxion $z\dot{s} + s\dot{z} - a\dot{t} = 0$, that is (reſtoring the Values of s, $\dot{s}$, $\dot{t}$) $z\ddot{y} + \dot{y}\dot{z} - a\ddot{x} = 0$, the ſame as before.

Ex. 14.

Suppoſe $2x\dot{x}\ddot{z} = a\dot{y}\ddot{y}$, and let $\dot{x}$ be a given or conſtant Quantity, then its Fluxion is $2\dot{x}^2\ddot{z} + 2x\dot{x}\dddot{z} = a\ddot{y}\ddot{y} + a\dot{y}\dddot{y}$.

Or thus, let $\dot{x} = b$, $\dot{z} = t$, $\ddot{y} = v$; then the given Equation becomes $2bx\dot{t} = a\dot{y}v$, whoſe Fluxion is $2bt\dot{x} + 2bx\dot{t} = av\dot{y} + ay\dot{v}$. And reſtoring $\dot{z}$, $\ddot{z}$, $\dot{x}$, $\dot{y}$, $\dddot{y}$ for t, $\dot{t}$, b, v, $\dot{v}$; you'll have $2\ddot{z}\dot{x}^2 + 2x\dot{x}\dddot{z} = a\ddot{y}\ddot{y} + a\dot{y}\dddot{y}$, the Fluxion required.

Ex. 15.

Let $\frac{a\dot{z}\ddot{x}}{\dot{x}^2} + \frac{y\dot{y}}{a} - \dot{y} = 0$, and ſuppoſe $\dot{y}$ conſtant; then its Fluxion is $\frac{a\dot{z}\dddot{x}}{\dot{x}^2} + \frac{a\ddot{z}\ddot{x}}{\dot{x}^2} - \frac{2a\dot{z}\ddot{x}^2}{\dot{x}^3} + \frac{\dot{y}^2}{a} = 0$.

Ex. 16.

Suppoſe $2a^3y\ddot{y} - x^2\dot{z}^2 - x^2\dot{z}\dot{x} = a^2\dot{y}^2 - \frac{a^3y^2\dddot{y}}{\dot{z}^2}$, and $\dot{z}$ invariable, its Fluxion will be $2a^3\dot{y}\ddot{y} + 2a^3y\dddot{y} - 2x\dot{x}\dot{z}^2 - 2x\dot{x}^2\dot{z} - x^2\dot{z}\ddot{x} = 2a^2\dot{y}\ddot{y} - \frac{2a^3y\dot{y}\dddot{y}}{\dot{z}^2} - \frac{a^3y^2\ddddot{y}}{\dot{z}^2}$.

Ex. 17.

Let $Y^3 = x$, where Y is the hyperbolic Log. of y; then its Fluxion is $3Y^2\dot{Y} = \dot{x}$; but $\dot{Y} = \frac{\dot{y}}{y}$, therefore $\dot{x} = \frac{3Y^2\dot{y}}{y}$.

Ex. 18.

Suppoſe $y^{x} = z$, then will $xY = Z$ (where Z, Y are the Hyp. Log. of z, y); the Fluxion of this Equation is $x\dot{Y} + Y\dot{x} = \dot{Z}$; but $\dot{Y} = \frac{\dot{y}}{y}$, and $\dot{Z} = \frac{\dot{z}}{z}$, whence $\frac{x\dot{y}}{y} + Y\dot{x} = \frac{\dot{z}}{z}$; therefore $\dot{z} = \left(\frac{xz\dot{y}}{y} + Yz\dot{x} =\right) xy^{x-1}\dot{y} + Yy^{x}\dot{x}$.

Ex. 19.

Let $y^{x^{v}} = z$; then $x^{v}Y = Z$, in Fluxions $x^{v}\dot{Y} + Y \times$ Fluxion of $x^{v} = \dot{z}$. But (by Ex. 18.) Fluxion of $x^{v} = vx^{v-1}\dot{x} + Xx^{v}\dot{v}$; and $\dot{Y} = \frac{\dot{y}}{y}$, alſo $\dot{Z} = \frac{\dot{z}}{z}$; therefore $\frac{x^{v}\dot{y}}{y} + Yvx^{v-1}\dot{x} + YXx^{v}\dot{v} = \frac{\dot{z}}{z} = \frac{\dot{z}}{y^{x^{v}}}$: Whence $\dot{z} = x^{v}y^{x^{v}-1}\dot{y} + vYy^{x^{v}}x^{v-1}\dot{x} + x^{v}YXy^{x^{v}}\dot{v}$.

PROP. IV.

Let the Fluent of $\overline{e+fz^{n}}^{\,m} z^{p}\dot{z} = \mathrm{A}$.

Fluent of $\overline{e+fz^{n}}^{\,m} z^{p+n}\dot{z} = \mathrm{B}$.

Fluent of $\overline{e+fz^{n}}^{\,m+1} z^{p}\dot{z} = \mathrm{P}$.

Then I ſay,

I. $\overline{p+1}\,.\,\mathrm{P} + \overline{m+1} \times nf\mathrm{B} = z^{p+1}\,\overline{e+fz^{n}}^{\,m+1}$.

II. $\mathrm{P} = e\mathrm{A} + f\mathrm{B}$.

DE-

DEMONSTRATION.

For I. $\overline{p+1}.\dot{\mathrm{P}}+\overline{m+1}.nf\dot{\mathrm{B}}=\overline{p+1}.z^p\dot{z}.\overline{e+fz^n}^{m+1}$
$+\overline{m+1}.nfz^{p+n}\dot{z}.\overline{e+fz^n}^{m}$
whoſe Fluent, by Cor. 2. Prop. I. is

$$\overline{p+1}\times\mathrm{P}+\overline{m+1}\times nf\mathrm{B}=z^{p+1}\overline{e+fz^n}^{m+1}.$$

II. $\overline{e+fz^n}^{m+1}z^p\dot{z}=\overline{e+fz^n}\times\overline{e+fz^n}^{m}z^p\dot{z}=$
$ez^p\dot{z}\times\overline{e+fz^n}^{m}+fz^{p+n}\dot{z}\times\overline{e+fz^n}^{m}$. That is
$\dot{\mathrm{P}}=e\dot{\mathrm{A}}+f\dot{\mathrm{B}}$, and the Fluent is $\mathrm{P}=e\mathrm{A}+f\mathrm{B}$.
Q. E. D.

COR. 1. Hence if any one of the Fluents A, B, P, be given, the other two will be found. Therefore,

COR. 2. $\mathrm{B}=\dfrac{z^{p+1}.\overline{e+fz^n}^{m+1}-\overline{p+1}\times e\mathrm{A}}{\overline{p+1+mn+n}\times f}.$

COR. 3. $\mathrm{A}=\dfrac{z^{p+1}\overline{e+fz^n}^{m+1}-\overline{p+1+mn+n}.f\mathrm{B}}{\overline{p+1}\times e}$

COR. 4. $\mathrm{P}=\dfrac{\overline{m+1}\times ne\mathrm{A}+z^{p+1}\overline{e+fz^n}^{m+1}}{p+1+mn+n}.$

COR. 5. $\mathrm{A}=\dfrac{\overline{p+1+mn+n}.\mathrm{P}-z^{p+1}\overline{e+fz^n}^{m+1}}{\overline{m+1}\times ne}.$

PROP. V.

Let $e+fz^n+gz^{2n}+hz^{3n}$ *&c.* $=\mathrm{V}$. $t=$ *Number of Terms in* V.
And the Fluent of $\mathrm{V}^m z^p\dot{z}=\mathrm{A}$.
Fluent of $\mathrm{V}^m z^{p+n}\dot{z}=\mathrm{B}$.
Fluent of $\mathrm{V}^m z^{p+2n}\dot{z}=\mathrm{C}$.
Fluent of $\mathrm{V}^m z^{p+3n}\dot{z}=\mathrm{D}$, *&c. continued to* t *Quantities.*
And Fluent of $\mathrm{V}^{m+1}z^p\dot{z}=\mathrm{P}$. *Then will,*

I.

$$\left.\begin{array}{l}\text{I. }\overline{p+1}\,.\,eA+\overline{p+1+mn+n}\,.\,fB+\\ \overline{p+1+2mn+2n}\,.\,gC+\overline{p+1+3mn+3n}\,.\,hD\\ +,\ \&c.\ \text{continu'd to } t \text{ Terms.}\end{array}\right\}=z^{p+1}V^{m+1}.$$

II. $P=eA+fB+gC+hD+$ &c. to t Terms.

DEMONSTRATION.

Let $p+1=r$, $m+1=s$. And putting the firſt Equation into Fluxions, there is $re\dot{A}+\overline{r+sn}\times f\dot{B}+\overline{r+2sn}\times g\dot{C}+\overline{r+3sn}\times h\dot{D}$ &c. $=rz^{p}\dot{z}V^{m+1}+sz^{p+1}V^{m}\dot{V}$; that is (reſtoring the Values of V, $\dot{V}$, $\dot{A}$, $\dot{B}$, &c. and dividing by $z^{p}V^{m}\dot{z}$) $re+\overline{r+sn}\times fz^{n}+\overline{r+2sn}.gz^{2n}+\overline{r+3sn}.hz^{3n}$ &c. $=r.\overline{e+fz^{n}+gz^{2n}+hz^{3n}}$ &c. $+sz\times\overline{nfz^{n-1}+2ngz^{2n-1}+3nhz^{3n-1}}$ &c. where both Sides of the Equation being manifeſtly equal, 'tis evident the Quantities from whence they were derived, that is, the Fluents in Equation the 1ſt, muſt alſo be equal. After the ſame Manner is the ſecond Equation very eaſily demonſtrated. Q. E. D.

COR. Hence if $t-1$ of the Fluents P, A, B, C, &c. be given; the reſt will be found.

PROP. VI.

Put $e+fz^{n}=V$, $k+lz^{n}=Y$, *and*
the Fluent of $z^{p}\dot{z}V^{m}Y^{q}=A$.
Fluent of $z^{p+n}\dot{z}V^{m}Y^{q}=B$.
Fluent of $z^{p+2n}\dot{z}V^{m}Y^{q}=C$.
Alſo Fluent of $z^{p}\dot{z}V^{m+1}Y^{q}=P$.
Fluent of $z^{p+n}\dot{z}V^{m+1}Y^{q}=Q$.

Then it will be

$$\text{I.}\quad \left.\begin{array}{l} \overline{p+1}\times ekA \\ \left.\begin{array}{l} +\overline{p+1+mn+n}\times fk \\ +\overline{p+1+qn+n}\times el \end{array}\right\}B \\ +\overline{p+1+mn+qn+2n}\,.\,flC \end{array}\right\} = z^{p+1}V^{m+1}Y^{q+1}.$$

$$\text{II.}\quad P = eA + fB.$$

$$\text{III.}\quad Q = eB + fC.$$

DEMONSTRATION.

Let $p+1=r$, $m+1=s$, $q+1=t$; and putting the firſt Equation into Fluxions, we have $rek\dot{A}\left.\begin{array}{l} +\overline{r+sn}\times fk \\ +\overline{r+tn}\times el \end{array}\right\}\dot{B}+\overline{r+sn+tn}\times fl\dot{C} =$ $rz^{p}\dot{z}V^{m+1}Y^{q+1}+sz^{p+1}Y^{q+1}V^{m}\dot{V}+tz^{p+1}V^{m+1}Y^{q}\dot{Y}$. Then reſtoring the Values of $\dot{A}$, $\dot{B}$, $\dot{C}$, $\dot{V}$, $\dot{Y}$, and dividing by $z^{p}V^{m}Y^{q}$, and then expunging V, Y; we have $rek\left.\begin{array}{l} +\overline{r+sn}\times fk \\ +\overline{r+tn}\times el \end{array}\right\}z^{n}+\overline{r+sn+tn}\times flz^{2n}$ $= r\times\overline{e+fz^{n}}\times\overline{k+lz^{n}}+snfz^{n}\times\overline{k+lz^{n}}+tnlz^{n}\times$ $\overline{e+fz^{n}}$, an Equation whoſe two Sides (when reduced) being manifeſtly the ſame, argues the Equation from whence it was deduced to be a true Equation. And the ſame may be eaſily ſhown of the other two Equations. Q. E. D.

COR. Hence if any 2 of the Fluents A, B, C, P, Q be given; the reſt may be found.

PROP. VII.

Let $e+fz^{n}+gz^{2n}+hz^{3n}$ *&c.* $= V$.
$k+lz^{n}+rz^{2n}+sz^{3n}$ *&c.* $= Y$.
$t =$ *Number of Terms in* V, $\tau =$ *Number of Terms in* Y.

And the Fluent of $z^p\dot{z}V^mY^q = A$.

Fluent of $z^{p+n}\dot{z}V^mY^q = B$.

Fluent of $z^{p+2n}\dot{z}V^mY^q = C$.

Fluent of $z^{p+3n}\dot{z}V^mY^q = D$. *&c. continu'd to* $t+\tau-1$ *Quantities.*

Alſo Fluent of $z^p\dot{z}V^{m+1}Y^q = P$.

Fluent of $z^{p+n}\dot{z}V^{m+1}Y^q = Q$.

Fluent of $z^{p+2n}\dot{z}V^{m+1}Y^q = R$. *&c. continu'd to* τ *Quantities.*

Then I ſay,

$$\text{I. } \overline{p+1}\times ekA \left.\begin{matrix}+\overline{p+1+qn+n}\times el\\ +\overline{p+1+mn+n}\times fk\end{matrix}\right\}B.$$

$$\left.\begin{matrix}+\overline{p+1+2qn+2n}\times er\\ +\overline{p+1+mn+qn+2n}\,.\,fl\\ +\overline{p+1+2mn+2n}\times gk\end{matrix}\right\}C \left.\begin{matrix}+\overline{p+1+3qn+3n}\times es\\ +\overline{p+1+mn+2qn+3n}\,.\,fr\\ +\overline{p+1+2mn+qn+3n}\,.\,gl\\ +\overline{p+1+3mn+3n}\times hk\end{matrix}\right\}D$$

$+$ *&c.* (continued to $t+\tau-1$ Terms) $= z^{p+1}V^{m+1}Y^{q+1}$.

II. $P = eA + fB + gC + hD$ *&c.* to t Terms.
III. $Q = eB + fC + gD + hE$ *&c.*
IV. $R = eC + fD + gE + hF$ *&c.*
V. $S = eD + fE + gF + hG$ *&c.*
&c. continued to τ Equations.

This is demonſtrated like the foregoing.

Cor. Hence if $t+\tau-2$ of the Fluents, P, Q, R, A, B, C, *&c.* are given, the reſt will be found.

PROP. VIII.

Let $V = e + fz^n$. $Y = k + lz^n$. $X = s + tz^n$:

And the Fluent of $z^p\dot{z}V^mY^qX^r = A$.

Fluent of $z^{p+n}\dot{z}V^mY^qX^r = B$.

Fluent of $z^{p+2n}\dot{z}V^mY^qX^r = C$.

Fluent of $z^{p+3n}\dot{z}V^mY^qX^r = D$.

Alſo Fluent of $z^p\dot{z}V^{m+1}Y^qX^r = P$.

Fluent of $z^{p+n}\dot{z}V^{m+1}Y^qX^r = Q$.

Fluent of $z^{p+2n}\dot{z}V^{m+1}Y^qX^r = R$.

Then it will be

$$\text{I. } \overline{p+1}\times eksA \left.\begin{matrix} +\,\overline{p+1+mn+n}\times fks \\ +\,\overline{p+1+qn+n}\times els \\ +\,\overline{p+1+rn+n}\times ekt \end{matrix}\right\}B$$

$$\left.\begin{matrix} +\,\overline{p+1+mn+qn+2n}.fls \\ +\,\overline{p+1+mn+rn+2n}.fkt \\ +\,\overline{p+1+qn+rn+2n}.elt \end{matrix}\right\}C + \overline{p+1+mn+qn+rn+3n}$$

$$\times fltD = z^{p+1}V^{m+1}Y^{q+1}X^{r+1}.$$

II. $P = eA + fB$.

III. $Q = eB + fC$.

IV. $R = eC + fD$.

The Truth of this is ſhewn as the reſt.

COR. Hence if (any) three of the Fluents A, B, C, D, P, Q, R be given, all the reſt will be found.

SCHOLIUM.

In the five laſt Propoſitions it muſt be obſerved, that any of the Fluents A, B, C; P, Q, R, *&c.* vaniſhes out of the Equation when its Coefficient is 0: And therefore in ſuch a Caſe that Fluent cannot be found by any of theſe Propoſitions.

If any one ſhould aſk how theſe Propoſitions are found out; the Anſwer is, by the common Analyſis. For taking any Equation in theſe Propoſitions, or aſſuming any Equation among the Fluents A, B, C, P, Q, R, at Pleaſure, and affecting them with unknown Coefficients; then putting that Equation into Fluxions (ſubſtituting the Values of $\dot{A}$, $\dot{B}$, $\dot{P}$, *&c.*) and then comparing the homologous Terms, theſe Coefficients will eaſily be determin'd, if the Thing is poſſible. And thus you may find out other Propoſitions of this Kind.

PROP. IX.

To transform a given Fluxion into another more ſimple one.

RULE.

1. For any compound Quantity in the given Fluxion put a new Letter or variable Quantity, by Help of which expunge the other variable Quantity and its Fluxion out of the given fluxionary Quantity; ſo will you have a new Fluxion inſtead of the former. If this be not ſimple enough, aſſume another variable Quantity inſtead of any compound Quantity contain'd in this laſt, and expunge the former aſſum'd Quantity and its Fluxion. Proceed thus till the transform'd Fluxion be as ſimple as poſſible.

2. Sometimes a compound Fluxion may be diſſolved into ſeveral other ſimpler ones, thus; aſſume two or more ſuch fluxionary Quantities, as you conceive will, when duly reduc'd, make up the given Fluxion, and let them be affected with indetermin'd Coefficients; then reducing them to the Form of the

given Fluxion, theſe Coefficients will be eaſily determin'd by comparing the homologous Terms.

3. Or in any fluxionary Equation of x and y, ſubſtitute ſome compound Quantity for one of them, ſuch as zx, or $z^n x$, for y; or $x\sqrt{zz-1}$ for y, or $z\dot{x}$ for $\dot{y}$, or any ſuch like, that by Subſtitution will make it ſimpler. For the whole Deſign of transforming any Quantity into another is in order to make it more ſimple, that the Fluent may be the eaſier found.

Ex. 1.

Let $\overline{e+fz^n}^{\,m}\times\overline{g+hz^n}^{\,s}z^{rn-1}\dot{z}$ be propoſed. Aſſume $v=e+fz^n$, then $z^n=\frac{v-e}{f}$, and $z^{rn}=\frac{\overline{v-e}^{\,r}}{f^r}$, and $z^{rn-1}\dot{z}=\frac{\overline{v-e}^{\,r-1}\dot{v}}{nf^r}$, and $g+hz^n=\frac{fg-he+hv}{f}=\frac{p+hv}{f}$, putting $p=fg-eh$. Therefore the given Fluxion $\overline{e+fz^n}^{\,m}\times\overline{g+hz^n}^{\,s}z^{rn-1}\dot{z}=\frac{\overline{v-e}^{\,r-1}}{nf^{r+s}}\times\overline{p+hv}^{\,s}v^m\dot{v}$.

Ex. 2.

There is given $\frac{z^{rn-1}\dot{z}}{\overline{k+lz^n\times e+fz^n+gz^{2n}}}$; aſſume $v=k+lz^n$, then $z^n=\frac{v-k}{l}$, and $z^{rn}=\frac{\overline{v-k}^{\,r}}{l^r}$, $z^{rn-1}\dot{z}=\frac{\overline{v-k}^{\,r-1}}{nl^r}\dot{v}$; put $p=\sqrt{ff-4eg}$, and then $\frac{1}{e+fz^n+gz^{2n}}=\frac{g}{p\cdot\overline{\frac{f-p}{2}+gz^n}}-\frac{g}{p\cdot\overline{\frac{f+p}{2}+gz^n}}$

as will be evident by reducing the two Terms to a common Denominator. Therefore, putting $a = \frac{f-p}{2}l-gk$, and $b = \frac{f+p}{2}l-gk$, the given Fluxion becomes $\frac{g}{p} \times \frac{\overline{v-k}^{r-1}\, v^{-1}\dot{v}}{nl^{r-1} \times \overline{a+gv}} - \frac{\overline{v-k}^{r-1}\, gv^{-1}\dot{v}}{npl^{r-1} \times \overline{b+gv}}$.

Ex. 3.

Let $\frac{\overline{e+fz^n}^{m}\, z^{\lambda n-1}\dot{z}}{\overline{g+hz^n}^{m+r}} = \dot{\text{F}}$ be given. Put $v = g + hz^n$. Then $z^n = \frac{v-g}{h}$, and $e+fz^n = \frac{fv-p}{h}$ (putting $p = fg - eh$) and $z^{\lambda n-1}\dot{z} = \frac{\overline{v-g}|^{\lambda-1}\,\dot{v}}{nh^{\lambda}}$; whence $\dot{\text{F}} = \frac{\overline{fv-p}^{m} \times \overline{v-g}^{\lambda-1}\, v^{-m-r}\dot{v}}{nh^{\lambda+m}} = \frac{\overline{1-gv^{-1}}^{\lambda-1}}{nh^{m+\lambda}} \times \overline{f-pv^{-1}}^{m}\, v^{\lambda-r-1}\dot{v}$.

Again, let $f-pv^{-1} = y$; then $v^{\lambda-r-1}\dot{v} = \frac{\overline{f-y}^{r-\lambda-1}\,\dot{y}}{p^{r-\lambda}}$, and $1-gv^{-1} = \frac{gy-eh}{p}$; whence $\dot{\text{F}} = \frac{\overline{gy-eh}^{\lambda-1}}{nh^{m+\lambda}p^{r-1}} \times \overline{f-y}^{r-\lambda-1}\, y^m\dot{y}$.

Otherwiſe.

Since $\frac{\overline{1-gv^{-1}}^{\lambda-1}}{nh^{m+\lambda}} \times \overline{f-pv^{-1}}^{m}\, v^{\lambda-r-1}\dot{v}$ is (by firſt multiplying and then dividing, by $1-gv^{-1}$) $= \frac{\overline{1-gv^{-1}}^{\lambda}}{nh^{m+\lambda}} \times \overline{f-pv^{-1}}^{m}\, v^{\lambda-r-1}\dot{v} + \frac{g \cdot \overline{1-gv^{-1}}^{\lambda-1}}{nh^{m+\lambda}} \times \overline{f-pv^{-1}}^{m}\, v^{\lambda-r-2}\dot{v}$; therefore expunging v and it becomes $\dot{F} = \dfrac{\overline{gy - eh}^{\lambda}}{nh^{m+\lambda}p^{r}} \times \overline{f - y}^{r-\lambda-1} y^{m}\dot{y} + \dfrac{g.\overline{gy - eh}^{\lambda-1}}{nh^{m+\lambda}p^{r}}$ $\times \overline{f - y}|^{r-\lambda} y^{m}\dot{y}$, and $y = f - pv^{-1} = \dfrac{e + fz^{n}}{g + hz^{n}}h$.

Ex. 4.

Suppoſe $\overline{e + fz^{n} + gz^{2n}}^{m} z^{rn-1}\dot{z} = \dot{F}$, put $e + fz^{n} + gz^{2n} = v$, $p = \frac{1}{4}ff - eg$, then $z^{n} = \dfrac{-\frac{1}{2}f + \sqrt{p + gv}}{g}$, and $z^{rn-1}\dot{z} = \dfrac{\overline{-\frac{1}{2}f + \sqrt{p + gv}}^{r-1}}{g^{r}} \times \dfrac{g\dot{v}}{2n\sqrt{p + gv}}$; therefore $\dot{F} = \dfrac{\overline{-\frac{1}{2}f + \sqrt{p + gv}}^{r-1}}{2ng^{r-1}} \times \dfrac{v^{m}\dot{v}}{\sqrt{p + gv}}$. If this be not thought ſimple enough, ſuppoſe,

Again $x = \sqrt{p + gv}$, then $v = \dfrac{xx - p}{g}$, and $v^{m+1} = \dfrac{\overline{xx - p}^{m+1}}{g^{m+1}}$, and $v^{m}\dot{v} = \dfrac{\overline{xx - p}^{m}}{g^{m+1}} \times 2x\dot{x}$; therefore $\dot{F} = \dfrac{\overline{x - \frac{1}{2}f}^{r-1}}{ng^{m+r}} \times \overline{x^{2} - p}|^{m} \dot{x}$. Here $x = \sqrt{p + gv} = \frac{1}{2}f + gz^{n}$.

Ex. 5.

To transform the Fluxion $\dfrac{z^{\lambda n-1}\dot{z}}{e + fz^{n} + gz^{2n}}$, aſſume $\dfrac{gz^{\lambda n-1}\dot{z}}{ge + gfz^{n} + ggz^{2n}} = \dfrac{Az^{\lambda n-1}\dot{z}}{B + gz^{n}} + \dfrac{Cz^{\lambda n-1}\dot{z}}{D + gz^{n}} =$ (by reducing to a commmon Denominator)

$$\dfrac{\begin{matrix}+AD\\+BC\end{matrix} z^{\lambda n-1}\dot{z} \quad \begin{matrix}+Ag\\+Cg\end{matrix} z^{\lambda n+n-1}\dot{z}}{BD \begin{matrix}+Bg\\+Dg\end{matrix} z^{n} + ggz^{2n}}$$

; then comparing the

homologous Terms, $Ag+Cg=0$, $AD+BC=g$, $BD=ge$, $Bg+Dg=fg$; whence is had $C=-A$, $D+B=f$, $D-B=\sqrt{ff-4eg}=p$; and thence $A=\frac{g}{p}$, $B=\frac{f-p}{2}$, $C=\frac{-g}{p}$, $D=\frac{f+p}{2}$. And therefore,

$$\frac{z^{\lambda n-1}\dot{z}}{e+fz^{n}+gz^{2n}}=\frac{g}{p}\times\frac{z^{\lambda n-1}\dot{z}}{\frac{f-p}{2}+gz^{n}}-\frac{g}{p}\times\frac{z^{\lambda n-1}\dot{z}}{\frac{f+p}{2}+gz^{n}}.$$

Otherwiſe thus.

Put $z^{n}=y-\frac{f}{2g}$, then $z^{\lambda n-1}\dot{z}=\frac{\dot{y}}{n}\times\overline{y-\frac{f}{2g}}\Big|^{\lambda-1}$. Whence the fluxional Quantity propoſed becomes $\dfrac{\frac{1}{n}\times\overline{y-\frac{f}{2g}}\Big|^{\lambda-1}}{e-\frac{ff}{4g}+gyy}$, which will be in rational Terms when λ is an Integer.

Ex. 6.

Let $\frac{z^{\lambda n-1}\dot{z}}{e+fz^{n}+gz^{2n}}$ be given, where λ is half of an odd Number, and $4eg$ greater than ff.

Aſſume $\frac{gz^{\lambda n-1}\dot{z}}{eg+fgz^{n}+ggz^{2n}}=\frac{Az^{\lambda n-\frac{1}{2}n-1}\dot{z}}{C-Dz^{\frac{1}{2}n}+gz^{n}}-\frac{Bz^{\lambda n-\frac{1}{2}n-1}\dot{z}}{C+Dz^{\frac{1}{2}n}+gz^{n}}=$ (by Reduction)

$$\frac{\begin{matrix}+AC\\-BC\end{matrix}z^{\lambda n-\frac{1}{2}n-1}\dot{z}\begin{matrix}+AD\\+BD\end{matrix}z^{\lambda n-1}\dot{z}\begin{matrix}+Ag\\-Bg\end{matrix}z^{\lambda n+\frac{1}{2}n-1}\dot{z}}{CC\begin{matrix}+2Cg\\-DD\end{matrix}z^{n}+ggz^{2n}}.$$

Now comparing the homologous Terms, we have $AC-BC=0=Ag-Bg$, $AD+BD=g$, $CC=eg$,

$2Cg - DD = fg$. From whence we have $C = \sqrt{eg}$, $D = \sqrt{2g\sqrt{eg} - fg}$, $A = \frac{g}{2D} = B$. Whence $\frac{z^{\lambda n - 1}\dot{z}}{e + fz^n + gz^{2n}} = \frac{g}{2D} \times \frac{z^{\lambda n - \frac{1}{2}n - 1}\dot{z}}{\sqrt{eg} - Dz^{\frac{1}{2}n} + gz^n} - \frac{g}{2D} \times \frac{z^{\lambda n - \frac{1}{2}n - 1}\dot{z}}{\sqrt{eg} + Dz^{\frac{1}{2}n} + gz^n}$: both which Terms may be transform'd into others ſtill more ſimple, after the Manner of the fourth Example.

Ex. 7.

Let $\frac{z^{\lambda n - 1}\dot{z}}{\overline{k + lz^n} \times \overline{e + fz^n + gz^{2n}}}$ be propoſed, where λ is half an odd Number, and $4eg$ greater than ff. Aſſume for it $\frac{Az^{\lambda n - n - 1}\dot{z} + Bz^{\lambda n - 1}\dot{z}}{e + fz^n + gz^{2n}} + \frac{Dz^{\lambda n - n - 1}\dot{z}}{k + lz^n}$, or for Brevity Sake (dividing by $z^{\lambda n - n - 1}\dot{z}$) $\frac{z^n}{\overline{k + lz^n} \times \overline{e + fz^n + gz^{2n}}} = \frac{A + Bz^n}{e + fz^n + gz^{2n}} + \frac{D}{k + lz^n} =$ (by Reduction) $\frac{\begin{array}{lll} & +Alz^n & \\ +Ak & +Bk & +Blz^{2n} \\ +De & +Df & +Dg \end{array}}{\overline{k + lz^n} \times \overline{e + fz^n + gz^n}}$; and comparing the homologous Terms, $Ak + De = 0$, $Al + Bk + Df = 1$, $Bl + Dg = 0$, whence (putting $t = ell - fkl + gkk$) we have $D = \frac{-lk}{t}$, $A = \frac{el}{t}$, $B = \frac{gk}{t}$. Whence $\frac{z^{\lambda n - 1}\dot{z}}{\overline{k + lz^{2n}} . \overline{e + fz^n + gz^{2n}}} = \frac{elz^{\lambda n - n - 1}\dot{z} + gkz^{\lambda n - 1}\dot{z}}{t \times \overline{e + fz^n + gz^{2n}}} - \frac{lkz^{\lambda n - n - 1}\dot{z}}{t \times \overline{k + lz^n}}$; and the firſt Part may be tranſformed again into others more ſimple by the ſixth Example.

I ſhall here add ſome more Examples, to illuſtrate the Method of proceeding.

Ex. 8.

To transform the Fluxion $\frac{z^{\delta n-1}\dot{z}}{p+qz^n+z^{2n}:\times r+sz^n+z^{2n}}$; aſſume $\frac{1}{p+qz^n+z^{2n}:\times r+sz^n+z^{2n}} = \frac{A+Bz^n}{p+qz^n+z^{2n}} + \frac{C+Dz^n}{r+sz^n+z^{2n}}$. Then multiplying croſs ways as before, we get

$$\left.\begin{array}{llll} & Ar+Asz^n & +Az^{2n} & \\ & +Br & +Bs & +Bz^{3n} \\ +Cp & +Cq & +C & \\ & +Dp & +Dq & +D \end{array}\right\} = 1.$$

Hence we ſhall get $Ar+Cp=1$. $As+Br+Cq+Dp=0$, $A+Bs+C+Dq=0$, and $B+D=0$. Hence may be had the Values of A, B, C, D; and then the Fluxion becomes $\frac{Az^{\delta n-1}\dot{z}+Bz^{\delta n+n-1}\dot{z}}{p+qz^n+z^{2n}} + \frac{Cz^{\delta n-1}\dot{z}+Dz^{\delta n+n-1}\dot{z}}{r+sz^n+z^{2n}}$.

Ex. 9.

To transform the Fluxion $\frac{\dot{y}}{a^3+y^3}$. Put it $= \frac{A+By}{aa-ay+yy}\dot{y} + \frac{c\dot{y}}{a+y}$; then multiplying alternately, and equating the homologous Terms; there is found $A=\frac{2}{3a}$, $B=\frac{-1}{3aa}$, $C=\frac{1}{3aa}$. Whence $\frac{\dot{y}}{a^3+y^3} = \frac{1}{3aa} \times \frac{2a-y}{aa-ay+yy}\dot{y} + \frac{1}{3aa} \times \frac{\dot{y}}{a+y}$.

The firſt Term may be further transformed thus; put $y=z+\frac{1}{2}a$, and there comes out $\frac{1}{3aa} \times \frac{\frac{3}{2}a-z}{\frac{3}{4}aa+zz}\dot{z}$; therefore $\frac{\dot{y}}{a^3+y^3} = \frac{1}{3aa} \times \frac{\frac{3}{2}a-z}{\frac{3}{4}aa+zz}\dot{z} + \frac{1}{3aa} \times \frac{\dot{y}}{a+y}$.

Ex. 10.

To transform the Fluxion $\frac{dz^m\dot{z}}{1+az^{2n+1}}$; m, n being Integers. Suppoſe $az^{2n+1}=x^{2n+1}$; and the Fluxion becomes $\frac{d}{a^{\frac{m-1}{2n+1}}}\times\frac{x^m\dot{x}}{1+x^{2n+1}}$. To tranſform $\frac{x^m\dot{x}}{1+x^{2n+1}}$, put $1+x^{2n+1}=\overline{1+x}\times :1-x+xx-x^3+x^4-\ldots+x^{2n}$. and $1-x+x^2-x^3\ldots+x^{2n}=\overline{1+px+x^2}\times\overline{1+qx+xx}\times\overline{1+rx+xx}$, &c. The latter Side of the Equation being multiply'd, and compared with the homologous Terms of the firſt Side, will give the Coefficients p, q, r, &c. Then

Let $\frac{x^m\dot{x}}{1+x^{2n+1}}=\frac{x^m\dot{x}}{\overline{1+x}\times\overline{1+px+xx}\times\overline{1+qx+xx}\times\overline{1+rx+xx}}$ *&c.* $=\dot{x}\times$ into $\frac{a}{1+x}+\frac{b+cx}{1+px+xx}+\frac{d+ex}{1+qx+xx}+\frac{f+gx}{1+rx+xx}$ *&c.* This reduced to a common Denominator, and the homologous Terms of the Numerators compared, will determine the Quantities a, b, c, d, &c.

And the ſame Way the Fluxion of $\frac{x^m\dot{x}}{1-x^{2n}}$ is transformed, by putting $1-x^{2n}=\overline{1+x}\times :1-x+x^2-x^3+\ldots-x^{2n-1}=\overline{1+x}\times\overline{1+px+xx}\times\overline{1+qx+xx}\times\overline{1+rx+xx}$, *&c.* and proceeding in all Reſpects as before: where m, n are Integers.

Ex. 11.

To transform the Fluxion $\frac{x^m\dot{x}}{1-x^n}$, m, n being Integers. Put $1-x^n=\overline{1-x}\times :1+x+xx+x^3\ldots+x^{n-1}$,

x^{n-1}, and $1+x+x^2+x^3 \ldots + x^{n-1} = \overline{1+px+xx} \times \overline{1+qx+xx} \times \overline{1+rx+xx}$, &c. the latter Side of the Equation being multiply'd, and the homologous Terms of both Sides compared, there will be found the Coefficients p, q, r, &c. Then

Let $\frac{x^m\dot{x}}{1-x^n} = \frac{x^m\dot{x}}{\overline{1-x} \times \overline{1+px+xx} \times \overline{1+qx+xx},\ \&c.}$ $= \dot{x} \times$ into $\frac{a}{1-x} + \frac{b+cx}{1+px+xx} + \frac{d+ex}{1+qx+xx}$ &c. This being reduced to a common Denominator, and the homologous Terms of the Numerator compared, will determine the Coefficients a, b, c, d, &c.

Ex. 12.

To transform $\frac{x^m\dot{x}}{1-x^{2n}}$, m, n being Integers. Put $1-x^{2n} = \overline{1-xx} \times : 1+xx+x^4+ \ldots . x^{m-2}$, and $1+xx+x^4 \ldots . x^{2n-2} = \overline{1+px+xx} \times \overline{1+qx+xx}$ &c. which multiply'd, and the homologous Terms compared, give p, q, r, &c. Then let $\frac{x^m\dot{x}}{1-x^{2n}}$ $= \frac{x^m\dot{x}}{\overline{1-xx} \times \overline{1+px+xx} \times \overline{1+qx+xx}} = \dot{x} \times :$ $\frac{a}{1-xx} + \frac{b+cx}{1+px+xx} + \frac{d+ex}{1+qx+xx}$ &c. Which reduced to a common Denominator, and the homologous Terms of the Numerator compared, give a, b, c, &c.

And the Fluxion $\frac{x^m\dot{x}}{1+x^{4n+2}}$ is transform'd by putting $1+x^{4n+2} = \overline{1+xx} \times : 1-x^2+x^4-x^6 \ldots + x^{4n} = \overline{1+xx} \times \overline{1+px+xx} \times \overline{1+qx+xx}$, &c. and proceeding in all Reſpects as before.

Likewiſe $\frac{x^m\dot{x}}{1 - x^{4n+2}}$ is transformed, by aſſuming the ſame Series as the laſt; that is, $1 - x^{4n+2} = \overline{1+xx} \times : 1 - x^2 + x^4 \ldots - x^{4n-2} = \overline{1+xx} \times \overline{1+px+xx} \times \overline{1+qx+xx}$ &c. and proceeding the ſame Way.

Ex. 13.

To transform $\frac{z^{\frac{1}{3}n-1}\dot{z}}{e \pm fz^n + gz^{2n}}$, when $n = 3$, $4eg > ff$; put $\frac{e}{g} = R^6$, $\frac{f}{2gR^3} = b$, then we have $\frac{1}{g} \times \frac{\dot{z}}{R^6 \pm 2bR^3z^3 + z^6}$. Now if $A =$ Arch whoſe natural Coſine is b, $\frac{A}{3} = B$, $\frac{360}{3}$ or $120 = C$; and s, t, v be the natural Coſines of B, $B+C$, $B+2C$; all taken from the Tables; then Rb, Rs, Rt, Rv will be the Coſines to the radius R; then will $R^6 \pm 2bR^2z^3 + z^6 = \overline{RR - 2Rsz + zz} \times \overline{RR - 2tRz + zz} \times \overline{RR - 2Rvz + zz}$. Then the Fluxion is transfom'd to

$$\frac{\frac{1}{g} \times \dot{z}}{\overline{R^2 - 2Rsz + zz} \times \overline{R^2 - ztRz + zz} \times \overline{R^2 - 2vRz + zz}}.$$

Again, let this be put $= \frac{K + kz}{R^2 - 2Rsz + zz} + \frac{L + lz}{R^2 - 2Rtz + zz} + \frac{M + mz}{R^2 - 2Rvz + zz} : \times \frac{\dot{z}}{g}$; and the Quantities K, k, L, l, &c. will be found by bringing all the Terms to a common Denominator and equating the Coefficients of the homologous Terms. Thus if R be made $= 1$, and $q = -2Rs$, $r = -2Rt$, ſ $= -2Rv$, there will come out $K = \frac{qq-1}{\overline{q-r} \times \overline{q-s}}$, $k = \frac{q}{\overline{q-r} \times \overline{q-s}}$; $L = \frac{rr-1}{\overline{r-q} \times \overline{r-s}}$, $l = \frac{r}{\overline{r-q} \times \overline{r-s}}$; $M = \frac{ss-1}{\overline{s-q} \times \overline{s-r}}$, $m = \frac{s}{\overline{s-q} \times \overline{s-r}}$. In which R may be reſtored, and the Terms made homogeneous; and for that Purpoſe dividing the Values of K, L, M, by R^4, and k, l, m, by R^3.

Ex. 14.

To transform the Fluxional Quantity $\frac{z^{\frac{\delta}{\lambda}n-1}\dot{z}}{e \pm fz^n + z^{2n}}$; where δ, λ, n, are Integers, and ff leſs than $4e$. Put $z^n = x^\lambda$, and then the Fluxion becomes $\frac{\lambda}{n} \times \frac{x^{\delta-1}\dot{x}}{e \pm fx^\lambda + x^{2\lambda}}$.

Again, to transform $\frac{x^{\delta-1}\dot{x}}{e \pm fx^\lambda + x^{2\lambda}}$; let $e^{\frac{1}{2}\lambda} = R$, $\frac{f}{2R^\lambda} = b$, A = Arch whoſe nat. Coſign is $\mp b$, $\frac{A}{3} = B$, $\frac{360}{\lambda} = C$; and S, T, V, W, &c. the nat. Coſines of B, B+C, B+2C, B+3C, &c. to λ Terms; all taken from the Tables: Then Rb, RS, RT, &c. will be the Coſines to the Radius R, and then $e \pm fx^\lambda + x^{2\lambda}$ or $R^{2\lambda} \pm 2bR^\lambda x^\lambda + x^{2\lambda} = \overline{RR - 2RSx + xx} \times \overline{RR - 2RTx + xx} \times \overline{RR - 2RVx + xx}$, &c. to λ Terms. Therefore the Fluxion now becomes

$$\frac{x^{\delta-1}\dot{x}}{\overline{RR - 2RSx + xx} \times \overline{RR - 2RTx + xx} \times \text{\&c.}}$$

Laſtly, put this laſt $= \frac{K+kx}{RR-2RSx+xx} + \frac{L+lx}{RR-2RTx+xx}$ &c. × into $x^{\delta-1}\dot{x}$, or rather × into $\frac{x^{\delta-1}\dot{x}}{R^{2\lambda-2}}$; to make the Quantities homogeneous. Or, for Brevity's Sake, let $R=1$, $q=-2Rs$, $r=-2RT$, $s=-2RV$, $t=-2RW$, &c. Then

$$x^{\delta-1}\dot{x}\times\frac{1}{\overline{1+qx+xx}\times\overline{1+rx+xx}\times\overline{1+sx+xx},\ \&c.}=$$

$$x^{\delta-1}\dot{x}\times:\frac{K+kx}{1+qx+xx}+\frac{L+lx}{1+rx+xx}+\frac{M+mx}{1+sx+xx}$$

&c. Then, by bringing the Terms to a common Denominator, K, *k*, L, &c. will be determined, by comparing the Coefficients as uſual. And at laſt R may be reſtor'd, to make the Terms homogeneous, or of the ſame Order. When the Operation is performed it will be found that when $\lambda=2$, $K=\frac{q}{q-r}$, $k=\frac{1}{q-r}$. $L=\frac{r}{r-q}$, $l=\frac{1}{r-q}$.

If $\lambda=3$. $K=\frac{qq-1}{\overline{q-r}\times\overline{q-s}}$, $k=\frac{q}{\overline{q-r}\times\overline{q-s}}$.

$$L=\frac{rr-1}{\overline{r-q}\times\overline{r-s}},\ l=\frac{r}{\overline{r-q}\times\overline{r-s}}.$$

$$M=\frac{ss-1}{\overline{s-q}\times\overline{s-r}},\ m=\frac{s}{\overline{s-q}\times\overline{s-r}}.$$

As in the laſt Example.

If $\lambda=4$.

$$K=\frac{q^3-2q}{\overline{q-r}\times\overline{q-s}\times\overline{q-t}},\ k=\frac{qq-1}{\overline{q-r}\times\overline{q-s}\times\overline{q-t}}.$$

$$L=\frac{r^3-2r}{\overline{r-q}\times\overline{r-s}\times\overline{r-t}},\ l=\frac{rr-1}{\overline{r-q}\times\overline{r-s}\times\overline{r-t}}.$$

M =

$$\mathrm{M} = \frac{s^3 - 2s}{\overline{s-q} \times \overline{s-r} \times \overline{s-t}},\ m = \frac{ss - 1}{\overline{s-q} \times \overline{s-r} \times \overline{s-t}}.$$

$$\mathrm{N} = \frac{t^3 - 2t}{\overline{t-q} \times \overline{t-r} \times \overline{t-s}},\ n = \frac{tt - 1}{\overline{t-q} \times \overline{t-r} \times \overline{t-s}}.$$

If $\lambda = 5$.

$$\mathrm{K} = \frac{q^4 - 3q^2 + 1}{\overline{q-r} \times \overline{q-s} + \overline{q-t} \times \overline{q-v}},\ k = \frac{q^3 - 2q}{\overline{q-r} \times \overline{q-s} \times \overline{q-t} \times \overline{q-v}}.$$

$$\mathrm{L} = \frac{r^4 - 3r^2 + 1}{\overline{r-q} \times \overline{r-s} \times \overline{r-t} \times \overline{r-v}},\ l = \frac{r^3 - 2r}{\overline{r-q} \times \overline{r-s} \times \overline{r-t} \times \overline{r-v}}.$$

$$\mathrm{M} = \frac{s^4 - 3ss + 1}{\overline{s-q} \times \overline{s-r} \times \overline{s-t} \times \overline{s-v}},\ m = \frac{s^3 - 2s}{\overline{s-q} \times \overline{s-r} \times \overline{s-t} \times \overline{s-v}}.$$

$$\mathrm{N} = \frac{t^4 - 3tt + 1}{\overline{t-q} \times \overline{t-r} \times \overline{t-s} \times \overline{t-v}},\ n = \frac{t^3 - 2t}{\overline{t-q} \times \overline{t-r} \times \overline{t-s} \times \overline{t-v}}.$$

$$\mathrm{P} = \frac{v^4 - 3v^2 + 1}{\overline{v-q} \times \overline{v-r} \times \overline{v-s} \times \overline{v-t}},\ p = \frac{v^3 - 2v}{\overline{v-q} \times \overline{v-r} \times \overline{v-s} \times \overline{v-t}}.$$

And the Law of the Progreſſion is evident; in the Denominator 'tis plain: And the Numerator of any Quantity K is determined by this Series, $q^{\lambda-1} - \frac{\lambda-2}{1} q^{\lambda-3} + \frac{\lambda-3}{1} \times \frac{\lambda-4}{2} q^{\lambda-5} - \frac{\lambda-4}{1} \times \frac{\lambda-5}{2} \times \frac{\lambda-6}{3} q^{\lambda-7} +$ &c.

If any body would know the Reaſon why the given Trinomial Quantity is divided into as many ſimple Trinomials, as there are Unites in λ, as is done in this and the laſt Example; they may ſee it demonſtrated in Cor. 3. Prop. 46. of my Elements of Trigonometry.

Ex. 15.

To transform the Fluxion $x\dot{x} + ay\dot{x} + y\dot{y} = 0$. Put $y = zx$, and $\dot{y} = z\dot{x} + x\dot{z}$, then the Fluxion becomes $x\dot{x} + azx\dot{x} + z^2x\dot{x} + x^2z\dot{z} = 0$, whence $\dot{x} = \frac{-xz\dot{z}}{zz + az + 1}$, and $\frac{\dot{x}}{x} = \frac{-z\dot{z}}{zz + az + 1}$.

Ex. 16.

To transform $ax\dot{y}+\dot{x}\sqrt{xx+yy}=0$. Put $x=y\sqrt{zz-1}$, then $\dot{x}=\frac{zz\dot{y}-\dot{y}+yz\dot{z}}{\sqrt{zz-1}}$, and the Fluxion becomes $ay\dot{y}\sqrt{zz-1}+\frac{zz\dot{y}-\dot{y}+yz\dot{z}}{\sqrt{zz-1}}\times y\dot{z}=0$, then $az^2y\dot{y}-ay\dot{y}+z^3y\dot{y}-zy\dot{y}+y^2z^2\dot{z}=0$, therefore $\dot{y}=\frac{-y^2z^2\dot{z}}{ay\times\overline{zz-1}+\overline{z^3-z}\times y}=\frac{-yz^2\dot{z}}{a\times\overline{zz-1}+z^3-z}$, and $\frac{\dot{y}}{y}=\frac{-z^2\dot{z}}{a\times\overline{zz-1}+z\times\overline{zz-1}}=\frac{-z^2\dot{z}}{\overline{a+z}\times\overline{zz-1}}$, therefore $\frac{\dot{y}}{y}+\frac{z^2\dot{z}}{\overline{z+a}\times\overline{z+1}\times\overline{z-1}}=0$. And the laſt Term may be further transform'd by putting $\frac{\dot{z}}{\overline{z+a}\times\overline{z+1}\times\overline{z-1}}=\frac{A\dot{z}}{z+a}+\frac{B\dot{z}}{z+1}+\frac{C\dot{z}}{z-1}$, and finding A, B, C, as in Ex. 5.

Ex. 17.

Let $a\dot{y}^3=x\dot{x}\dot{y}^2+b\dot{x}\dot{y}^2-c\dot{x}^3$, be given; put $\dot{x}=z\dot{y}$, then $a\dot{y}^3=xz\dot{y}^3+bz\dot{y}^3-cz^3\dot{y}^3$, or $a=xz+bz-cz^3$, whence $x=\frac{cz^3-bz+a}{z}=cz^2-b+\frac{a}{z}$, and $\dot{x}=2cz\dot{z}-\frac{a\dot{z}}{zz}=z\dot{y}$, whence $\dot{y}=2c\dot{z}-\frac{a\dot{z}}{z^3}$, &c.

Ex. 18.

To transform the Fluxion $\frac{\dot{x}}{\sqrt[n]{ax^p+bx^n}}$; put $ax^p+bx^n=x^nz^n$, then $z^n-b=ax^{p-n}$, and $\frac{\text{Log.}}{z^n-b}$

$\overline{z^n-b}$ = Log. $a + \overline{p-n} \times$ Log. x. In Fluxions $\frac{nz^{n-1}\dot{z}}{\overline{z^n-b}} = \frac{p-n}{x}\dot{x}$, hence $\frac{\dot{x}}{\sqrt[n]{\overline{ax^p+bx^n}}} = \frac{\dot{x}}{xz}$ $= \frac{n}{p-n} \times \frac{z^{n-2}\dot{z}}{\overline{z^n-b}}$.

Ex. 19.

To transform the Fluxion $\overline{vx\dot{x}-xx\dot{v}} \times \text{Fl.}\ \frac{v^2\dot{x}}{x} = c\dot{x}$; let $v = x^n$, and $\dot{v} = nx^{n-1}\dot{x}$; then will $\overline{x^{n+1}\dot{x} - nx^{n+1}\dot{x}} \times \frac{x^{2n}}{2n} = c\dot{x}$, or $\frac{1-n}{2n}x^{3n+1}\dot{x} = c\dot{x}$, then comparing the Indexes and Coefficients on both Sides, $3n+1=0$, and $\frac{1-n}{2n} = c$; which gives $n = -\frac{1}{3}$, and $c = -2$. Therefore $v = \frac{1}{\sqrt[3]{x}}$.

PROP. X.

An Equation being given containing the Fluxions of Quantities; to find the Fluents, either in ſimple Terms, or in a Series thereof proceeding ad infinitum.

Rules.

When the *Fluxions* are *not* of the *ſame Order* in all the Terms, ſupply the Defect by the Powers of ſome *given Fluxion* ſuppoſed to be Unity: *Fractional* and *Compound Quantities* muſt be *reduced* by Multiplication, Diviſion, *&c.* Radical Quantities (except ſuch where the fluxionary Part is in a given Ratio to the

Fluxion of the Root) muſt be *reduced* to ſimple Terms by *Involution*; and the Roots of *adfected Equations* muſt be *extracted.* This Preparation being made,

I.

1. If the Equation then can be ſo order'd, that *every Term* has only *one variable Quantity* and its *Fluxion*; multiply ſeparately *each Term* by its *variable Quantity*, and then *divide* it by the *Fluxion* and its *new Index.*

2. If any Term be ſuch a *Radical Quantity* that the fluxionary Part may be divided by the Fluxion of the *Root* (or Part under the Vinculum); and that by ſuch Diviſion the Quotient may be a given Quantity. *Multiply* that Term by the ſaid *Root*, and then divide by the Fluxion of the *Root* and the new *Index*, for the Fluent of that radical Quantity.

3. If any Term be divided by the *firſt Power* of the variable Quantity; then the Fluent of that Term muſt be found by itſelf thus; multiply the *given Coefficient* and the Number 2.302585 into the *Logarithm* of that variable Quantity, for the *Fluent* of that Term. Or thus by Series, ſubſtitute for this *variable Quantity* the Sum or Difference of ſome *given Quantity* and another *variable Quantity*, and its Fluxion for the Fluxion; then this new Term being actually *divided*, the Fluent will be found as in the firſt Article.

4. If in one Side of an Equation there be *two Terms* containing *two variable Quantities*, each multiply'd into the Fluxion of the other; then by Art. 1. find the Fluent of either Term, ſuppoſing only one Quantity variable, and this will be the Fluent of both theſe Terms. And if there be *three Terms* containing *three variable Quantities*, where the Fluxion of each is multiply'd into the Product of the other two; then find the Fluent of any one of theſe Terms (by Art. 1.) conſidering the other Quantities as given; and this will be the Fluent of all three.

5. Laſtly, all the *Terms* being *collected* on the correſponding Sides of the Equation, will give an Equation containing the *Relation* of the variable Quantities.

EXAMPLE I.

Let $\dot{y} = d\dot{x} - \dot{x}$, then the Fluent $y = dx - x$.

Ex. 2.

Let $\dot{x} = ay^2\dot{y}$, the Fluent is $x = \frac{ay^3}{3}$.

Ex. 3.

Let $\dot{y} = \frac{-x\dot{x}}{\sqrt{aa - x\dot{x}}}$, or $\dot{y} = \overline{aa - xx}|^{-\frac{1}{2}} \times - x\dot{x}$; multiply by the Root $aa - xx$, and there is $\overline{aa - xx}|^{\frac{1}{2}} \times - x\dot{x}$; divide this by $-2x\dot{x} \times \frac{1}{2}$, and there ariſes $\overline{aa - xx}|^{\frac{1}{2}}$ or $\sqrt{aa - xx} = y$.

Ex. 4.

Suppoſe $\dot{y} = \frac{a\dot{x}}{x}$, then $y = 2.302585a \times \text{Log. } x$. Or thus, let $b + z = x$, and $\dot{z} = \dot{x}$, then will $\dot{y} = \frac{a\dot{z}}{b+z} = \frac{a\dot{z}}{b} - \frac{az\dot{z}}{bb} + \frac{az^2\dot{z}}{b^3} - \frac{az^3\dot{z}}{b^4} + \text{\&c.}$ and the Fluent $y = \frac{az}{b} - \frac{az^2}{2b^2} + \frac{az^3}{3b^3} - \frac{az^4}{4b^4}$ &c.

Ex. 5.

Let $\dot{z} = 2ay^3x\dot{x} + 3ax^2y^2\dot{y}$, where there are the Quantities y^3 and ax^2, and one multiply'd into the other's Fluxion; the Fluent of $2ay^3x\dot{x}$ (ſuppoſing y invariable) is ay^3x^2, therefore $z = ay^3x^2$.

Ex. 6.

$\dot{v} = ax^m\dot{x}$, then $v = \frac{ax^{m+1}}{m+1}$.

Ex. 7.

Suppoſe $\dot{y} = \overline{a\dot{x} + x\dot{x}}\,\sqrt{2ax+xx}$, then $y = \dfrac{\overline{a\dot{x} + x\dot{x}} \times \overline{2ax+xx}|^{\frac{3}{2}}}{\frac{3}{2} \times 2a\dot{x} + 2x\dot{x}} = \dfrac{\overline{2ax+xx}|^{\frac{3}{2}}}{3}$.

Ex. 8.

Let $\dot{v} = \dfrac{\dot{x}}{y} - \dfrac{x\dot{y}}{yy}$, that is $\dot{v} = y^{-1}\dot{x} - xy^{-2}\dot{y}$, and the Fluent $v = xy^{-1} = \dfrac{x}{y}$.

Ex. 9.

Let $\dot{v} = y^2z^3\dot{x} + 2xz^3y\dot{y} + 3xy^2z^2\dot{z}$; then $v = xy^2z^3$.

Ex. 10.

Suppoſe $\dot{y} = \dfrac{\dot{x}}{x^3} - \dfrac{\dot{x}}{x^2} + \dfrac{a\dot{x}}{x^{\frac{1}{2}}} - x^{\frac{1}{2}}\dot{x} + x^{\frac{3}{2}}\dot{x}$; multiply the firſt Side by $\dfrac{y}{\dot{y}}$, and the laſt by $\dfrac{x}{\dot{x}}$; and there will be $y = \dfrac{1}{xx} - \dfrac{1}{x} + ax^{\frac{1}{2}} - x^{\frac{3}{2}} + x^{\frac{5}{2}}$; then divide by the Index of the Power in each Term, and the Fluent will be $y = \dfrac{-1}{2xx} + \dfrac{1}{x} + 2ax^{\frac{1}{2}} - \frac{2}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}}$.

Ex. 11.

Let the given Equation be $\dot{x} = \dfrac{2bbc\dot{y}}{\sqrt{ay^3}} + \dfrac{3y^2\dot{y}}{a+b} + \dot{y}\sqrt{by+cy}$, that is $\dot{x} = \dfrac{2bbc}{\sqrt{a}} \times y^{-\frac{3}{2}}\dot{y} + \dfrac{3}{a+b} \times y^2\dot{y} + y^{\frac{1}{2}}\dot{y}\sqrt{b+c}$; the Fluent is $x = \dfrac{-4bbc}{\sqrt{a}}y^{-\frac{1}{2}} + \dfrac{y^3}{a+b} + \frac{2}{3}y^{\frac{3}{2}}\sqrt{b+c}$.

Ex.

Ex. 12.

Let $\dot{y} = x^{-\frac{1}{2}}\dot{x} - x^{\frac{1}{2}}\dot{x} + \frac{\dot{x}}{x}$; the Fluent will be $y = 2x^{\frac{1}{2}} - \frac{2}{3}x^{\frac{3}{2}} + 2.302585 \text{ Log. } x$. Or thus, put $a - z = x$, and $-\dot{z} = \dot{x}$, then $\frac{\dot{x}}{x} = \frac{-\dot{z}}{a-z} = -\frac{\dot{z}}{a} - \frac{z\dot{z}}{aa} - \frac{z^2\dot{z}}{a^3}$ — &c. whence $\dot{y} = x^{-\frac{1}{2}}\dot{x} - x^{\frac{1}{2}}\dot{x} - \frac{\dot{z}}{a} - \frac{z\dot{z}}{a^2} - \frac{z^2\dot{z}}{a^3}$ &c. and then $y = 2x^{\frac{1}{2}} - \frac{2}{3}x^{\frac{3}{2}} - \frac{z}{a} - \frac{z^2}{2aa} - \frac{z^3}{3a^3}$ — &c.

Ex. 13.

Let the Equation be $\dot{y} = \overline{\alpha + \beta z^{\eta}}^{\mu} z^{\pi}\dot{z}$; involve that radical Quantity, and then $\dot{y} = \alpha^{\mu}z^{\pi}\dot{z} + \mu\alpha^{\mu-1}\beta z^{\eta+\pi}\dot{z} + \mu \times \frac{\mu-1}{2}\alpha^{\mu-2}\beta^2 z^{2\eta+\pi}\dot{z}$ + &c. whence the Fluent is $y = \frac{\alpha^{\mu}z^{\pi+1}}{\pi+1} + \frac{\mu\alpha^{\mu-1}\beta z^{\eta+\pi+1}}{\eta+\pi+1} + \mu \times \frac{\mu-1}{2} \times \frac{\alpha^{\mu-2}\beta^2 z^{2\eta+\pi+1}}{2\eta+\pi+1}$ + &c.

Ex. 14.

Suppoſe $\dot{y} = \theta e z^{\theta-1}\dot{z} + \genfrac{}{}{0pt}{}{\theta f}{+\mu\eta f} z^{\theta+\eta-1}\dot{z}$ into $\overline{e + fz^{\eta}}^{\mu-1}$, that is $\dot{y} = \theta z^{\theta-1}\dot{z} \times \overline{e + fz^{\eta}}^{\mu} + \mu\eta f z^{\theta+\eta-1}\dot{z} \times \overline{e + fz^{\eta}}^{\mu-1}$. Here are the two Quantities z^{θ} and $\overline{e + fz^{\eta}}^{\mu}$ each multiply'd into the other's Fluxion; therefore $y = z^{\theta} \times \overline{e + fz^{\eta}}^{\mu}$.

Ex. 15.

Let $\dot{y} = \theta e z^{\theta-1}\dot{z} + \begin{matrix}\theta f \\ +\mu n f\end{matrix} z^{\theta+n-1}\dot{z} + \begin{matrix}\theta g \\ +2\mu n g\end{matrix} z^{\theta+2n-1}\dot{z}$ into $\overline{e+fz^{n}+gz^{2n}}|^{\mu-1}$, that is $\dot{y} = \theta z^{\theta-1}\dot{z}\times\overline{e+fz^{n}+gz^{2n}}|^{\mu} + \overline{\mu f n z^{\theta+n-1}\dot{z}+2\mu n g z^{\theta+2n-1}\dot{z}}\times\overline{e+fz^{n}+gz^{2n}}|^{\mu-1}$. Here are two Quantities z^{θ} and $\overline{e+fz^{n}+gz^{2n}}|^{\mu}$ each multiply'd into the Fluxion of the other, therefore $y = z^{\theta}\times\overline{e+fz^{n}+gz^{2n}}|^{\mu}$, the Fluent.

Ex. 16.

Suppoſe $\dot{y} = \theta e g z^{\theta-1}\dot{z} + \begin{matrix}\overline{\theta+\mu n}\times fg \\ +\overline{\theta+rn}\times eh\end{matrix} z^{\theta+n-1}\dot{z} + \overline{\theta+\mu n+rn}\times fhz^{\theta+2n-1}\dot{z}$ into $\overline{e+fz^{n}}|^{\mu-1}\times\overline{g+hz^{n}}|^{r-1}$, that is $\dot{y} = \theta z^{\theta-1}\dot{z}\times\overline{e+fz^{n}}|^{\mu}\times\overline{g+hz^{n}}|^{r} + \mu n f z^{\theta+n-1}\dot{z}\times\overline{g+hz^{n}}|^{r}.\overline{e+fz^{n}}|^{\mu-1} + rnhz^{\theta+n-1}\dot{z}.\overline{e+fz^{n}}|^{\mu}.\overline{g+hz^{n}}|^{r-1}$. Now here are the 3 Quantities z^{θ}, $\overline{e+fz^{n}}^{\mu}$ and $\overline{g+hz^{n}}^{r}$, and the Fluxion of each is multiply'd into the Product of the reſt; therefore the Fluent is $y = z^{\theta}\times\overline{e+fz^{n}}|^{\mu}\times\overline{g+hz^{n}}|^{r}$.

Ex. 17.

Let this Eqution be given $\dot{y}^{2} = \dot{x}\dot{y} + x^{2}\dot{x}^{2}$; by extracting the Root, $\dot{y} = \frac{1}{2}\dot{x} \pm \dot{x}\sqrt{\frac{1}{4}+xx} =$ (by Involution) $\frac{1}{2}\dot{x} + \frac{1}{3}\dot{x} + x^{2}\dot{x} - x^{4}\dot{x} + 2x^{6}\dot{x}$, &c. or $\dot{y} = \frac{1}{2}\dot{x} - \frac{1}{2}\dot{x} - x^{2}\dot{x} + x^{4}\dot{x} - 2x^{6}\dot{x}$, &c. Whence $y = x + \frac{x^{3}}{3} - \frac{x^{5}}{5} + \frac{2x^{7}}{7}$ &c. or $y = \frac{-x^{3}}{3} + \frac{x^{5}}{5} - \frac{2x^{7}}{7}$ &c.

Ex. 18.

Suppoſe $\dot{y}^3 + ax\dot{x}^2\dot{y} + a^2\dot{x}^2\dot{y} - x^3\dot{x}^3 - 2a^3\dot{x}^3 = 0$. Extracting the Root of this Equation, $\dot{y} = a\dot{x} - \frac{x\dot{x}}{4} + \frac{x^2\dot{x}}{64a} + \frac{131x^3\dot{x}}{512a^2}$, &c. then the Fluent will be $y = ax - \frac{x^2}{8} + \frac{x^3}{192a} + \frac{131x^4}{2048a^2} +$ &c.

Ex. 19.

Suppoſe this Equation $\dot{z}^3 - c\dot{z}^2 - 2x^2\dot{z} - c^2\dot{z} + 2x^3 + c^3 = 0$. Put $\dot{x} = 1$, by which multiply the deficient Terms, and then you have $\dot{z}^3 - c\dot{z}^2\dot{x} - 2x^2\dot{x}^2\dot{z} - c^2\dot{x}^2\dot{z} + 2x^3\dot{x}^3 + c^3\dot{x}^3 = 0$. And extracting the Root, which will be threefold, then $\dot{z} = c\dot{x} + x\dot{x} - \frac{x^2\dot{x}}{4c} + \frac{x^3\dot{x}}{32cc}$, &c. Or $\dot{z} = c\dot{x} - x\dot{x} + \frac{3x^2\dot{x}}{4c} + \frac{15x^3\dot{x}}{32cc}$, &c. Or, laſtly, $\dot{z} = -c\dot{x} - \frac{x^2\dot{x}}{2c} - \frac{x^3\dot{x}}{2cc} + \frac{x^5\dot{x}}{4c^4}$ &c. Whence the Fluent will either be $z = cx + \frac{1}{2}xx - \frac{x^3}{12c} + \frac{x^4}{128cc}$, &c. or $z = cx - \frac{1}{2}x^2 + \frac{x^3}{4c} + \frac{15x^4}{128cc}$, &c. or $z = -cx - \frac{x^3}{6c} - \frac{x^4}{8cc} + \frac{x^6}{24c^4}$, &c.

Ex. 20.

Let $\frac{a^4\dot{y}^2}{y} - 4a^3\dot{y}^2 + 4aay\dot{y}^2 + \frac{aaxx}{y}\dot{y}^2 = 4ax^2\dot{y}^2 + \frac{x^2\dot{x}^2}{y} - 4x^2y\dot{y}^2$; tranſpoſe $4ax^2\dot{y}^2 - 4x^2y\dot{y}^2$ and then multiply by y, and you have $a^4\dot{y}^2 - 4a^3y\dot{y}^2 + 4a^2y^2\dot{y}^2 + aax^2\dot{y}^2 - 4ax^2y\dot{y}^2 + 4y^2x^2\dot{y}^2 = x^2\dot{x}^2$; divide

by

by $aa+xx$, and there ariſes $aa\dot y^2 - 4ay\dot y^2 + 4y^2\dot y^2 = \frac{x^2\dot x^2}{aa+xx}$, extract the Root, and $a\dot y - 2y\dot y = \frac{x\dot x}{\sqrt{aa+xx}}$; whence the Fluent is $ay - yy = \pm\sqrt{aa+xx}$.

Ex. 21.

Let $\dot z\dot y + z\ddot y = a\ddot x$; then its Fluent is $z\dot y = a\dot x$.

Ex. 22.

Suppoſe $a\dot y\dot x^2 - b\dot y\dot x^2 = bx\dot x\ddot y + b\dot x^3$, or $a\dot y\dot x^2 - b\dot y\dot x^2 - bx\dot x\ddot y = b\dot x^3$, ſuppoſing $\dot x$ invariable; then its Fluent will be $ay\dot x^2 - bx\dot x\dot y = bx\dot x^2$.

Ex. 23.

Let the Equation be $az\dot z^2 + aa\dot z\dot y - ab\dot z\dot x = bc\dot z^2 - abc\ddot x$; then the Fluent is $\frac{az^2\dot z}{2} + aay\dot z - abx\dot z = bcz\dot z - abc\dot x$; ſuppoſing $\dot z$ invariable.

II.

If the Quantities cannot be ſo ſeparated but one or both of the *Fluxions* contained in the Equation will be affected with *both* the *variable Quantities*; then reduce the Equation ſo that one of the Fluxions alone, (or at moſt, only affected with its own flowing Quantity) may poſſeſs one Side of the Equation, and the other Fluxion affected with both the variable Quantities be on the other Side, in ſimple Terms. Then

1. Range all the Quantities that are on the ſecond Side of the Equation ſo, that all the fluxional Quantities affected only with its *own Fluent*, may ſtand *horizontally* at Top, proceeding regularly according to the Indices either increaſing or decreaſing, ac-

cording as the Fluent is to be had in an aſcending or deſcending Series; and all the Terms affected with the other flowing Quantity may ſtand *perpendicularly* on the *left* Hand according to their Indices.

2. Begin at the left Hand, and find the Fluent of the *firſt* Term of the *horizontal* Row, for the firſt Term of the Fluent; then ſubſtitute this inſtead of the other variable Quantity in all the Terms of the *perpendicular* Row, writing their new Values over-againſt them under their proper Indices in the *horizontal* Row, then proceed to find the ſecond, third, *&c.* Term of the Fluent; by ſumming up all the fluxionary Quantities of the ſame Index into one Term, and then finding its Fluent; all which Terms of the Fluent are to be gradually ſubſtituted for the Powers of the other Quantity as you go along.

3. And this Operation may be performed various Ways, by *aſſuming* any given Quantity for the firſt Term, or perhaps for ſome other Term of the Fluent, and oftentimes it will be *neceſſary* to do ſo. If any of the Terms be divided by the *firſt Power* of its flowing Quantity, it will ſometimes be neceſſary to ſubſtitute for this Quantity, the Sum or Difference of a given Quantity and another variable Quantity; and then reduce the Terms to the preſcribed Form. And this is the *Newtonian Rule* for finding the *Fluent*.

Ex. 24.

Let $a\dot{y} + x\dot{y} - a\dot{x} - x\dot{x} = y\dot{x} + \frac{ay^3\dot{x} + xy^3\dot{x}}{a^3} - \frac{ax^2\dot{x} + x^3\dot{x}}{a^2}$, to find y. Divide by $a+x$ and reduce the Equation, then $\dot{y} = \dot{x} + \frac{y\dot{x}}{a+x} + \frac{y^3\dot{x}}{a^3} - \frac{x^2\dot{x}}{a^2}$ $= \dot{x} - \frac{x^2\dot{x}}{aa} + \frac{y^3\dot{x}}{a^3} + \frac{y\dot{x}}{a} - \frac{yx\dot{x}}{aa} + \frac{yx^2\dot{x}}{a^3}$ *&c.* Then the Work will be as follows.

	$\dot{x}$	*	$-\frac{x^2\dot{x}}{aa}$	
$+\frac{y\dot{x}}{a}$		$+\frac{x\dot{x}}{a}$	$+\frac{x^2\dot{x}}{2aa}$	$-\frac{x^3\dot{x}}{2a^3}$, &c.
$-\frac{yx\dot{x}}{aa}$			$-\frac{x^2\dot{x}}{aa}$	$-\frac{x^3\dot{x}}{2a^3}$, &c.
$+\frac{yx^2\dot{x}}{a^3}$				$+\frac{x^3\dot{x}}{a^3}$, &c.
&c.				
$+\frac{y^3\dot{x}}{a^3}$				$+\frac{x^3\dot{x}}{a^3}$, &c.

$$\dot{y} = \dot{x} + \frac{x\dot{x}}{a} - \frac{3x^2\dot{x}}{2aa} + \frac{x^3\dot{x}}{a^3} \text{ \&c.}$$

$$y = x + \frac{x^2}{2a} - \frac{x^3}{2aa} + \frac{x^4}{4a^3} \text{ \&c.} = \text{Fluent.}$$

In this Example I write $\dot{x} - \frac{x^2\dot{x}}{aa}$ horizontally, and $\frac{y\dot{x}}{a} - \frac{yx\dot{x}}{aa} + \frac{yx^2\dot{x}}{a^3}$ &c. $+ \frac{y^3\dot{x}}{a^3}$ perpendicularly; then I bring down $\dot{x}$ into the Value of $\dot{y}$, and get its Fluent x, and put it into the Value of y. Then I ſubſtitute x for y in each Term of the perpendicular Row, placing their reſpective Values $\frac{x\dot{x}}{a}$, $-\frac{x^2\dot{x}}{aa}$, $+\frac{x^3x}{a^3}$, and $+\frac{x^3\dot{x}}{a^3}$ againſt each, under the proper Powers of x.

Then I bring down the Term in the ſecond Place $+\frac{x\dot{x}}{a}$, and write its Fluent $\frac{x^2}{2a}$ in the Value of y; then I write $\frac{x^2}{2a}$ for y in the Terms $+\frac{y\dot{x}}{a} - \frac{yx\dot{x}}{aa}$, &c. and the Reſult $\frac{x^2\dot{x}}{2a^2}$, $-\frac{x^3\dot{x}}{2a^3}$ I put over-againſt them as before. Then I take the Sum of the Terms in the third Place $-\frac{x^2\dot{x}}{aa}+\frac{x^2\dot{x}}{2aa}-\frac{x^2\dot{x}}{aa}$ and ſet this Sum $-\frac{3x^2\dot{x}}{2aa}$ underneath, and get its Fluent $-\frac{x^3}{2aa}$, which I ſubſtitute for y as before. And thus I proceed as far as I pleaſe.

Ex. 25.

Let $\dot{y}=\dot{x}-3x\dot{x}+y\dot{x}+x^2\dot{x}+yx\dot{x}$, to find y.

Here x the firſt Term being found and written for y, you get $-2x\dot{x}$, and its Fluent $-xx$ for the ſecond Term, which being alſo written for y, you'll get + [illegible]$^2\dot{x}$, and its Fluent $\frac{x^3}{3}$ for the third Term; and ſo on as below.

	$+\dot{x}-3x\dot{x}+x^2\dot{x}$
$+y\dot{x}$	$+x\dot{x}-x^2\dot{x}+\frac{1}{3}x^3\dot{x}-\frac{1}{6}x^4\dot{x}$ &c.
$+yx\dot{x}$	$+x^2\dot{x}-x^3\dot{x}+\frac{1}{3}x^4\dot{x}$ &c.
$\dot{y}=$	$\dot{x}-2x\dot{x}+x^2\dot{x}-\frac{2}{3}x^3\dot{x}+\frac{1}{6}x^4\dot{x}$ &c.
$y=$	$x-xx+\frac{1}{3}x^3-\frac{1}{6}x^4+\frac{1}{30}x^5$ &c.

Otherwiſe thus.

Here I take a for the firſt Term of y, and writing it inſtead of y, and get $ax+x$ the next Term, which being again written for y, I get the third Term ax^2-x^2, and ſo on; ſee the Work.

+

	$+\dot{x}-3x\dot{x}+x^2\dot{x}$
$+y\dot{x}$	$a\dot{x}\ \substack{+a\\+1}x\dot{x}\ \substack{+a\\-1}x^2\dot{x}\ \substack{+\frac{2}{3}a\\+\frac{1}{3}}x^3\dot{x}$ &c.
$+yx\dot{x}$	$+ax\dot{x}\ \substack{+a\\+1}x^2\dot{x}\ \substack{+a\\-1}x^3\dot{x}$ &c.
$\dot{y}=$	$\substack{+a\\+1}\dot{x}\ \substack{+2a\\-2}x\dot{x}\ \substack{+2a\\+1}x^2\dot{x}\ \substack{+\frac{5}{3}a\\-\frac{2}{3}}x^3\dot{x}$ &c.
$y=$	$a\ \substack{+a\\+1}x\ \substack{+a\\-1}x^2\ \substack{+\frac{2}{3}a\\+\frac{1}{3}}x^3\ \substack{+\frac{5}{12}a\\-\frac{1}{6}}x^4$ &c.

Ex. 26.

Suppoſe this Equation $\dot{y}=-3x\dot{x}+3yx\dot{x}+y^2\dot{x}-y^2x\dot{x}+y^3\dot{x}-y^3x\dot{x}$ &c. $+6yx^2\dot{x}-6x^2\dot{x}+8yx^3\dot{x}-8x^3\dot{x}+10yx^4\dot{x}-10x^4\dot{x}$ &c. To find the Value of y as far as 7 Dimenſions of x.

I place the Terms in Order according to the following Table, and then I work as before; and moreover I ſubjoin the Square and Cube of the Value of y gradually produced, to be ſubſtituted by Degrees into their proper Places towards the right Hand, in the Values of the Marginals on the left, as follows.

	$-3x\dot{x}-6x^2\dot{x}-8x^3\dot{x}-10x^4x-12x^5\dot{x}-14x^6\dot{x}$ &c.
$+3yx\dot{x}$	$-\frac{9}{2}x^3\dot{x}-6x^4\dot{x}-\frac{75}{8}x^5\dot{x}-\frac{273}{20}x^6\dot{x}$ &c.
$+6yx^2\dot{x}$	$-9x^4\dot{x}-12x^5\dot{x}-\frac{75}{4}x^6\dot{x}$ &c.
$+8yx^3\dot{x}$	$-12x^5\dot{x}-16x^6\dot{x}$ &c.
$+10yx^4\dot{x}$	$-15x^6\dot{x}$ &c.
&c.	
$+yy\dot{x}$	$+\frac{9}{4}x^4\dot{x}+6x^5\dot{x}+\frac{107}{8}x^6\dot{x}$ &c.
$-yx^2\dot{x}$	$-\frac{9}{4}x^5\dot{x}-6x^6\dot{x}$ &c.
&c.	
$+y^3\dot{x}$	$-\frac{27}{8}x^6\dot{x}$ &c.

$$\dot{y}=-3x\dot{x}-6x^2\dot{x}-\frac{25}{2}x^3\dot{x}-\frac{91}{4}x^4\dot{x}-\frac{333}{8}x^5\dot{x}-\frac{367}{5}x^6\dot{x} \text{ \&c.}$$

$$y=-\frac{3}{2}x^2-2x^3-\frac{25}{8}x^4-\frac{91}{20}x^5-\frac{111}{16}x^6-\frac{367}{35}x^7 \text{ \&c.}$$

$$y^2=\frac{9}{4}x^4+6x^5+\frac{107}{8}x^6 \text{ \&c.}$$

$$y^3=-\frac{27}{8}x^6 \text{ \&c.}$$

Ex. 27.

Let $\dot{y}=\frac{y\dot{x}}{xx}+\frac{\dot{x}}{xx}+3\dot{x}+2x\dot{x}-\frac{4\dot{x}}{x}$; to find y in a deſcending Series. Here the Terms in the horizontal Row muſt be placed to proceed from the greater Indices to the leſſer: And the Work will be as below.

	$+2x\dot{x}+3\dot{x}-\frac{4\dot{x}}{x}+\frac{\dot{x}}{xx}$
$+\frac{y\dot{x}}{xx}$	$\dot{x}+\frac{4\dot{x}}{x} * -\frac{\dot{x}}{x^3}+\frac{\dot{x}}{2x^4}$ &c.

$$\dot{y}=2x\dot{x}+4\dot{x}+0+\frac{\dot{x}}{xx}-\frac{\dot{x}}{x^3}+\frac{\dot{x}}{2x^4} \text{ \&c.}$$

$$y=x^2+4x \quad * -\frac{1}{x}+\frac{1}{2x^2}-\frac{1}{6x^3} \text{ \&c.}$$

Here obſerve, that any given Quantity might have been inſerted between the Terms $4x$ and $-\frac{1}{x}$; and ſo y might be extracted an infinite Variety of Ways. But if that fluxionary Term had not vaniſhed, then we had been obliged to ſubſtitute $a+z$ or $a-z$ for x in the given Equation before it could be reſolved.

Ex. 28.

Suppoſe the given Equation $cx^2\dot{x}+y\dot{x}=a\dot{y}$, by Reduction $\dot{y}=\frac{cx^2\dot{x}}{a}+\frac{y\dot{x}}{a}$; then the Work will be very eaſily performed as in the following Table.

	$\frac{cx^2\dot{x}}{a}$			
$+\frac{y\dot{x}}{a}$		$+\frac{cx^3\dot{x}}{3aa}$	$+\frac{cx^4\dot{x}}{3.4a^3}$	$+\frac{cx^5\dot{x}}{3.4.5a^4}$
$\dot{y}=$	$\frac{cx^2\dot{x}}{a}$	$+\frac{cx^3\dot{x}}{3aa}$	$+\frac{cx^4\dot{x}}{3.4a^3}$	$+\frac{cx^5\dot{x}}{3.4.5a^4}$
$y=$	$\frac{cx^3}{3a}$	$+\frac{cx^4}{3.4aa}$	$+\frac{cx^5}{3.4.5a^3}$	$+\frac{cx^6}{3.4.5.6a^4}$ &c.

Therefore $y=2caa\times:\frac{x^3}{2.3a^3}+\frac{x^4}{2.3.4a^4}+\frac{x^5}{2.3.4.5a^5}$ &c. But (by Sch. II. Prob. 2. Sect. II.) $1+\frac{x}{a}+\frac{x^2}{2aa}+\frac{x^3}{2.3a^3}+\frac{x^4}{2.3.4a^4}$ &c. $=$ Number of the Hyp. Log. $\frac{x}{a}$. Therefore $y=2caa\times$ Num. of the Hyp. Log. $\frac{x}{a}-2caa\times:1+\frac{x}{a}+\frac{xx}{2aa}:$

And by the ſame Rule the Fluent of $x^2\dot{y}=x\dot{x}-\dot{x}-yx^2\dot{x}$, will be found $y=\frac{1}{x}$.

III.

III.

When the given Equation contains firſt Fluxions alone, or if it contains *firſt*, *ſecond*, or *third* &c. *Fluxions*, as in the following Example, where $2ax\dot{y}^2 - ay^2\ddot{x} + 2x^2\dot{y}^2 - 2y^2\dot{x}^2 = 0$, to find x expreſſed by y and given Quantities; it will be reſolved by the following general Method.

1. Make the Equation $= 0$; and *aſſume* an *indetermin'd Series* to repreſent the Series required; as $x = Ay^n + By^{n+r} + Cy^{n+s} + Dy^{n+t}$, &c. wherein the Indices n, $n+r$, $n+s$ continually increaſe if y be very ſmall, or decreaſe if it be great.

2. To find the firſt Index n; ſuſtitute into the given Equation the *firſt Term* Ay^n, its *Fluxion*, and *ſecond Fluxion*, &c. inſtead of x, $\dot{x}$, and $\ddot{x}$, &c. (if they be there) and then you'll have a new Equation, as $2aAy^n - \overline{nn - n} \times aAy^n + 2A^2y^{2n} - 2n^2A^2y^{2n} = 0$, ſuppoſing $\dot{y} = 1$. Make two (or more) of the Terms *equal* to nothing that have the *leaſt Indices* equal to one another, for an *aſcending Series*; or thoſe that have the *greateſt Indices* equal, for a *deſcending Series*; then by equating their Indices or elſe their Coefficients, n will be found. Or if there happen to be only *one Term* with ſuch leaſt or greateſt Index, make its *Coefficient* $= 0$, which will deſtroy that Term, and perhaps give the Value of n: Thus, in this Example for an aſcending Series, you'll have $2aAy^n - \overline{nn - n} \times aAy^n = 0$, or $nn - n = 2$, whence $n = 2$.

3. For the other Indices; *ſubſtitute* the Value of n into the foregoing Equation, and you'll have $2aAy^2 - 2aAy^2 + 2A^2y^4 - 8A^2y^4 = 0$. Then take the *leaſt Index* in an *aſcending Series*, or the *greateſt* in a *deſcending* one from each of the reſt; and find all the

poſſible Numbers that reſult by *adding* all theſe Remainders to themſelves and to one another as oft as poſſible, and then you have r, s, t, &c. here $4-2=2$ only one Remainder; then $2, 4, 6$, *&c.* $= r, s, t$, &c. And therefore the Series is $Ay^2+By^4+Cy^6+Dy^3$ &c. $=x$.

4. For determining the Coefficients A, B, C, *&c.* *ſubſtitute* into the given Equation the *Values* of x, $\dot{x}$, $\ddot{x}$, *&c.* expreſſed by the foregoing Series $Ay^2+By^4+Cy^6$, &c. and put the *Sum* of the *Coefficients* of every ſeveral Power of y *equal* to nothing, and thence A, B, C, *&c.* will be gradually found. Thus $\dot{x}=2Ay+4By^3+6Cy^5$, &c. $\ddot{x}=2A+12By^2+30Cy^4$, &c. where $y=1$, theſe being ſubſtituted, the Work will be perform'd as below.

5. If the firſt Equation for the Coefficients be an *adfected* Equation containing ſeveral *different Powers* of A, then that Equation will afford ſeveral Roots or Values of A; and as many different Roots ſo many different Series may be obtain'd. And if A has ſeveral equal Values or Roots in this Equation, then you muſt divide the leaſt Remainder above by that Number (of equal Roots of A, one of which you aſſume for its Value;) then proceed as in Art. 3. taking this Quotient for another Remainder.

6. If a Series be required to be expreſs'd in *Terms* of that *Quantity* whoſe 2d, 3d Fluxion, *&c.* is in the Equation; it muſt firſt be got in Terms of the *other* Quantity that has no ſecond, third, *&c.* Fluxion; and then the Series reverted.

SCHOL. As there are fluxionary Equations that admit of ſeveral Solutions, and may have the Root expreſs'd various ways; ſo there are Equations that cannot be reſolved at all as being impoſſible; and others that are very difficult to be reſolved, and may require the utmoſt Skill of the Analiſt, and ſometimes a different Proceſs from theſe Rules.

Ex.

Ex. 29.

Let $2ax\dot{y}^2 - ay^2\ddot{x} + 2x^2\dot{y}^2 - 2y^2\dot{x}^2 = 0$; to find x expreſſed by y and given Quantities.

The Form of the Series found by the foregoing Rule is $x = Ay^2 + By^4 + Cy^6 + Dy^3$ &c. This and its Fluxions being ſubſtituted into the given Equation will be as follows.

$$\begin{array}{r|rrrrl}
+2ax\dot{y}^2 & +2aAy^2 & +2aBy^4 & +2aCy^6 & +2aDy^3 & \text{\&c.} \\
-ay^2\ddot{x} & -2aAy^2 & -12aBy^4 & -30aCy^6 & -56aDy^8 & \text{\&c.} \\
+2x^2\dot{y}^2 & & +2A^2y^4 & +4ABy^6 & \begin{matrix}+2BB\\+4AC\end{matrix}y^8 & \text{\&c.} \\
-2y^2\dot{x}^2 & & -8A^2y^4 & -32ABy^6 & \begin{matrix}-32BB\\-48AC\end{matrix}y^8 & \text{\&c.}
\end{array}$$

Then equating their reſpective Coefficients; $2aA - 2aA = 0$, therefore A may be taken at Pleaſure. Again $-10aB - 6A^2 = 0$, thence $B = \frac{-3A^2}{5a}$. After the ſame Manner $C = \frac{3A^3}{5aa}$, and $D = -\frac{31A^4}{45a^3}$ &c. Whence $x = Ay^2 - \frac{3A^2}{5a}y^4 + \frac{3A^3}{5aa}y^6 - \frac{31A^4}{45a^3}y^8$ &c.

Otherwiſe thus for a deſcending Series.

Make the Terms $2A^2y^{2n} - 2n^2A^2y^{2n} = 0$, or $1 - nn = 0$, whence $n = 1$; this ſubſtituted for n will produce $2aAy + 2A^2y^2 - 2A^2y^2 = 0$. Take the greateſt Index 2 from the reſt, which is 1, and there remains -1; therefore r, s, t, &c. are -1, -2, -3, &c. and the Series is $Ay + B + Cy^{-1} + Dy^{-2}$ &c. and the Operation will be as follows.

$$\begin{array}{l|lllll}
+2ax\dot{y}^2 & & +2aAy & +2aB & +2aCy^{-1} & \&c. \\
-ay^2\ddot{x} & & & & -2aCy^{-1} & \&c. \\
+2x^2\dot{y}^2 & +2A^2y^2 & +4ABy & \begin{matrix}+2BB\\+4AC\end{matrix} & \begin{matrix}+4BC\\+4AD\end{matrix}y^{-1} & \&c. \\
-2y^2\dot{x}^2 & -2A^2y^2 & & +4AC & +8ADy^{-1} & \&c.
\end{array}$$

$$\text{or}\left\{\begin{matrix}+2A^2\\-2A^2\end{matrix}y^2 \begin{matrix}+2aA\\+4AB\end{matrix}y \begin{matrix}+2aB\\+2BB\\+8AC\end{matrix} \begin{matrix}+4BC\\+12AD\end{matrix}y^{-1}\ \&c. = 0.\right.$$

Hence, equating the Coefficients; $2A^2 - 2A^2 = 0$; and A may be again taken at Pleaſure; likewiſe $B = \frac{-a}{2}$, $C = \frac{aa}{16A}$, $D = \frac{a^3}{96A^2}$, &c. whence the Series is known, and $x = Ay - \frac{a}{2} + \frac{aa}{16Ay} + \frac{a^3}{96A^2y^2}$, &c.

Ex. 30.

Let $a^3\dddot{y} - a^2\dot{y}\ddot{y} + y\dot{y}\dot{x}^2 = 0$, to find y in an aſcending Series.

Aſſume $y = Ax^n + Bx^{n+r} + Cx^{n+s}$, &c. and ſubſtituting the firſt Term and its Fluxions for y and its Fluxions, we have $n \times \overline{n-1} \times \overline{n-2} \times a^2Ax^{n-3} - n^2 \times \overline{n-1} \times a^2A^2x^{2n-3} + nA^2x^{2n-1} = 0$. Since the Index $n-3$ is the leaſt Index, make its Coefficient $n \times \overline{n-1} \times \overline{n-2} = 0$, and take one of the Roots $n = 2$. Whence the Indices will be -1, 1, 3, ſubtract the leaſt, -1 from the reſt, and there remains 2, 4; therefore r, s, $t = 2$, 4, 6 &c. whence $y = Ax^2 + Bx^4 + Cx^6$ &c. and the reſt of the Work is as follows.

$$\begin{array}{l|llll}
+a^3\dddot{y} & +24a^3Bx & +120a^3Cx^3 & +336a^3Dx^5 & \&c. \\
-a^2\dot{y}\ddot{y} & -4a^2A^2x & -32a^2ABx^3 & -72a^2ACx^5 & \&c. \\
 & & & -48a^2BB & \\
+y\dot{y}\dot{x}^2 & & +2A^2x^3 & +6ABx^5 & \&c.
\end{array}$$

Hence $24a^3B = 4a^2A^2$, and A may be taken at Pleaſure; let $A = \frac{1}{2a}$, then $B = \frac{1}{24a^3}$, alſo $C = \frac{1}{720a^5}$, $D = \frac{+1}{40320a^7}$ &c. Whence $y = \frac{x^2}{2a} + \frac{x^4}{24a^3} + \frac{x^6}{720a^5} + \frac{x^8}{40320a^7}$ &c.

Otherwiſe for a deſcending Series.

Let $2n-1$ and $n-3$ be ſuppoſed to be the greateſt Indices, then $n-3 = 2n-1$, and $n = -2$, and theſe Indices will be -5, -7; and taking -5 from -7 the Remainder is -2; therefore r, s, $t = -2$, -4, -6, &c. Wherefore $y = Ax^{-2} + Bx^{-4} + Cx^{-6}$, &c. the reſt of the Work will be thus,

$$\begin{array}{l|llll}
+a^3\dddot{y} & -24a^3Ax^{-5} & -120a^3Bx^{-7} & -336a^3Cx^{-9} & \&c. \\
-a^2\dot{y}\ddot{y} & & +12a^2A^2x^{-7} & +64a^2ABx^{-9} & \&c. \\
+y\dot{y}\dot{x}^2 & -2A^2x^{-5} & -6ABx^{-7} & \begin{matrix}-8AC \\ -4BB\end{matrix}\,x^{-9} & \&c.
\end{array}$$

Therefore $A = -12a^3$, $B = 36a^5$, $C = -\frac{1368}{10}a^7$ &c.

And $y = -12a^3x^{-2} + 36a^5x^{-4} - \frac{1368}{10}a^7x^{-6}$ &c.

Ex. 31.

Let the Equation be $\dot{x} = \frac{1}{2}y\dot{y} - 4y^2\dot{y} + 2x^{\frac{1}{2}}y\dot{y} - \frac{4}{5}x^2\dot{y} + 7y^{\frac{5}{2}}\dot{y} + 2y^3\dot{y}$, to find x in y aſcending.

Let $x = Ay^{n} + By^{n+r} + Cy^{n+s}$ *&c*; and ſubſtituting Ay^{n} and its Fluxion for x and $\dot{x}$, there ariſes $-nAy^{n-1} + \frac{1}{2}y - 4y^{2} + A^{\frac{1}{2}}y^{\frac{n}{2}+1} - \frac{4}{5}A^{2}y^{2n} + 7y^{\frac{5}{2}} + 2y^{3} = 0$. Suppoſe $n-1$ and 1 to be the leaſt Indices, and you'll have $-nAy^{n-1} + \frac{1}{2}y = 0$, and $n-1 = 1$, or $n = 2$, and all the Indices will be 1, 2, 4, $2\frac{1}{2}$, 3. Take 1 from the Reſt and the Remainders are 1, $1\frac{1}{2}$, 2, 3; whence r, s, t *&c.* are 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, *&c.* and $x = Ay^{2} + By^{3} + Cy^{3\frac{1}{2}} + Dy^{4}$, *&c.* hence

$$\begin{array}{r|lllll}
-\dot{x} & -2Ay & -3By^{2} & -3\frac{1}{2}Cy^{2\frac{1}{2}} & -4Dy^{3} & \&c. \\
+\frac{1}{2}y & +\frac{1}{2}y & & & & \\
-4y^{2} & & -4y^{2} & & & \\
+2yx^{\frac{1}{2}} & & +2A^{\frac{1}{2}}y^{2} & * & +\dfrac{By^{3}}{A^{\frac{1}{2}}} & \\
-\frac{4}{5}x^{2} & & * & * & * & \\
+7y^{\frac{5}{2}} & & & +7y^{\frac{5}{2}} & * & \\
+2y^{3} & & & & +2y^{3} &
\end{array}$$

therefore $A = \frac{1}{4}$, likewiſe $B = -1$, $C = 2$, $D = 0$, *&c.* and the Series is $x = \frac{1}{4}y^{2} - y^{3} + 2y^{\frac{7}{2}}$, *&c.*

Ex. 32.

Let $\dot{z}^{3} - c\dot{z}^{2} - 2x^{2}\dot{z} - c^{2}\dot{z} + 2x^{3} + c^{3} = 0$, to find z in a Series of x aſcending. Put $z = Ax^{n} + Bx^{n+r} + Cx^{n+s}$ *&c.* and by Subſtitution according to the Rule, we have $n^{3}A^{3}x^{3n-3} - cn^{2}A^{2}x^{2n-2} - 2nAx^{n+1} - nccAx^{n-1} + 2x^{3} + c^{3} = 0$; and ſuppoſing the leaſt Indices $n-1$ and 0 to be equal, we have $n = 1$. And all theſe Indices will become 0, 2, 3: And ſubſtracting the leaſt 0, the Remainders will be 2, 3; whence r, s, t, &c. $= 2, 3, 4,$ *&c.* and the Series is $z = Ax + Bx^{3} + Cx^{4} + Dx^{5}$ *&c.* and the Operation as follows,

$$\begin{array}{l|lllll}
+\dot{z}^3 & +A^3 & +9A^2Bx^2 & +12A^2Cx^3 & \begin{matrix}+15A^2D\\+27AB^2\end{matrix}x^4 & \&c. \\
-c\dot{z}^2 & -cA^2 & -6cABx^2 & -8cACx^3 & \begin{matrix}-9cB^2\\-10cAD\end{matrix}x^4 & \&c. \\
-2x^2\dot{z} & & -2Ax^2 & * & -6Bx^4 & \&c. \\
-c^2\dot{z} & -c^2A & -3c^2Bx^2 & -4c^2Cx^3 & -5c^2Dx^4 & \&c. \\
+2x^3 & & & +2x^3 & * & \\
+c^3 & +c^3 & & & &
\end{array}$$

Here $A^3-cA^2-c^2A+c^3=0$, and $A=+c$ or $-c$: If $A=+c$, then B will be infinite; therefore $A=-c$, and then $B=-\frac{1}{6c}$, $C=-\frac{1}{8cc}$, $D=0$, &c. And $z=-cx-\frac{x^3}{6c}-\frac{x^4}{8cc}$ &c.

If you take $A=+c$; then ſince the Equation $A^3-cA^2-c^2A+c^3=0$, contains two Roots $=c$; therefore divide 2 (the leaſt Remainder) by 2 (the Number of equal Roots) and the Quotient is 1. Then by Help of the Remainders 1, 2, 3, &c. you'll get the Series $z=Ax+Bx^2+Cx^3$, &c. with this proceed as with the former, and you will get other Series for the Value of z, wherein B may be taken at Pleaſure.

Ex. 33.

Suppoſe $e\dot{y}+fz^m\dot{y}+dyz^{m-1}\dot{z}=z^p\dot{z}$, to find y. Aſſume $y=Az^n+Bz^{n+r}+Cz^{n+s}$ &c. then putting Az^n and its Fluxion for y and $\dot{y}$; and you have $enAz^{n-1}+fnAz^{m+n-1}+dAz^{m+n-1}-z^p=0$; take the Indices $n-1=p$, then $n=p+1$, and the Indices become p, $p+m$, and the common Difference $=m$, and $r, s, t=m, 2m, 3m$, &c. Whence $y=Az^{p+1}+Bz^{p+m+1}+Cz^{p+2m+1}$, &c. therefore

$+e\dot{y}$

$$\begin{array}{r|llll}
+e\dot{y} = & \overline{p+1}.eAz^{p} & +\overline{p+1+m}.eBz^{p+m} & +\overline{p+1+2m}.eCz^{p+2m} & \&c. \\
+fz^{m}\dot{y} = & & +\overline{p+1}.fAz^{p+m} & +\overline{p+1+m}.fBz^{p+2m} & \&c. \\
+dyz^{m-1}\dot{z} = & & +\quad dAz^{p+m} & +\quad dBz^{p+2m} & \&c. \\
-z^{p}\dot{z} = & -z^{p} & & &
\end{array}$$

Hence $A = \frac{1}{\overline{p+1}.e}$, $B = -\frac{d+\overline{p+1}.f}{\overline{p+1+m}.e}A$, $C = -\frac{d+\overline{p+1+m}.f}{\overline{p+1+2m}.e}B$, &c. and putting $\varepsilon = p+1$, and $\delta = \frac{d}{f}+\overline{p+1}$, then $y = \frac{z^{\varepsilon}}{\varepsilon e} - \frac{\delta f}{\overline{\varepsilon+m}.e}Az^{\varepsilon+m} - \frac{\overline{\delta+m}.f}{\overline{\varepsilon+2m}.e}Bz^{\varepsilon+2m} - \frac{\delta+2m.f}{\varepsilon+3m.e}Cz^{\varepsilon+3m}$ &c.

Ex. 34.

Let $\dot{x} = \overline{\alpha+\beta z^{m}}^{\mu} z^{\pi}\dot{z}$, to find x by a Series of z. Suppoſe $x = y \times \overline{\alpha+\beta z^{m}}^{\mu}$, this in Fluxions gives $\dot{x} = \dot{y} \times \overline{\alpha+\beta z^{m}}^{\mu} + \mu m\beta yz^{m-1}\dot{z} \times \overline{\alpha+\beta z^{m}}^{\mu-1} = \overline{\alpha+\beta z^{m}}^{\mu} z^{\pi}\dot{z}$, that is $\alpha\dot{y} + \beta z^{m}\dot{y} + \mu m\beta yz^{m-1}\dot{z} - \alpha z^{\pi}\dot{z} - \beta z^{\pi+m}\dot{z} = 0$. Let $y = Az^{n} + Bz^{n+r} + Cz^{n+s}$ &c. then by Subſtitution $\alpha nAz^{n-1} + n\beta Az^{n+m-1} + \mu m\beta Az^{n+m-1} - \alpha z^{\pi} - \beta z^{\pi+m} = 0$. And putting the Indices $n-1 = \pi$, then $n = \pi+1$, and theſe Indices are π, $\pi+m$, and the common Difference m, whence $y = Az^{\pi+1} + Bz^{\pi+1+m} + Cz^{\pi+1+2m}$ &c.

$$\begin{array}{lr|llll}
\text{And} & \alpha\dot{y} & \overline{\pi+1}.\alpha Az^{\pi} & +\overline{\pi+1+m}.\alpha Bz^{\pi+m} & +\overline{\pi+1+2m}.\alpha Cz^{\pi+2m} & \&c. \\
& +\beta z^{m}\dot{y} & & +\overline{\pi+1}.\beta A & +\overline{\pi+1+m}.\beta B & \&c. \\
& +\mu m\beta yz^{n-1}\dot{y} & & +\mu m\beta A & +\mu m\beta B & \&c. \\
& -\alpha z^{\pi}\dot{z}-\beta z^{\pi+m}\dot{z} & -\alpha & -\beta & &
\end{array}$$

Hence $A = \frac{1}{\pi+1}$, $B = -\frac{\mu m\beta}{\overline{\pi+1}\times\overline{\pi+1+m}\times\alpha}$, $C = \frac{\overline{\mu m+\pi+1+m}\times\beta^2\mu m}{\overline{\pi+1}\times\overline{\pi+1+m}\times\overline{\pi+1+2m}\times\alpha^2}$ &c. whence putting $q = \frac{\beta z^m}{\alpha}$, $\varepsilon = \pi+1$, $\delta = \varepsilon+\mu m$. P, Q, R, each preceding Term with its Sign, then $x = \overline{\alpha+\beta z^m}^{\mu} \times : \frac{z^\varepsilon}{\varepsilon} - \frac{\mu m q}{\varepsilon+m} P - \frac{\overline{\delta+m}.q}{\varepsilon+2m} Q - \frac{\overline{\delta+2m}.q}{\varepsilon+3m} R -$ &c. $=$ Fl: $\overline{\alpha+\beta z^n}^{\mu} z^\pi \dot{z}$

Ex. 35.

Let $\dot{x} = \overline{e+fz^n+gz^{2n}+hz^{3n}\ \&c.}\times\overline{\alpha+\beta z^n+\gamma z^{2n}+\delta z^{3n}\ \&c.}^{\mu} \times z^\pi\dot{z}$, to find x by z. Suppose $x = y.\overline{\alpha+\beta z^n+\gamma z^{2n}+\delta z^{3n}\ \&c.}^{\mu+1}$ this put into Fluxions, and then the whole divided by $\overline{\alpha+\beta z^n+\gamma z^{2n}}^{\mu}$ &c. you have $z^\pi\dot{z}\times\overline{e+fz^n+gz^{2n}\ \&c.} = \dot{y}\times\overline{\alpha+\beta z^n+\gamma z^{2n}\ \&c.} + \overline{\mu+1}.y\times\overline{n\beta z^{n-1}+2n\gamma z^{2n-1}\ \&c.}$ Let $y = Az^m + Bz^{m+r} + Cz^{m+s}$ &c. then Az^m and its Fluxion being substituted for y and $\dot{y}$, for the first Term of each compound Quantity in the foregoing Equation, and there will be $-ez^\pi + m\alpha Az^{m-1} + \overline{\mu+1}\times n\beta Az^{m+n-1}$ &c. $= 0$; and making the Indices π and $m-1$ equal, then $m = \pi+1$, and the other Index is $\pi+1+n$, and the common Difference n, whence $y = Az^{\pi+1} + Bz^{\pi+1+n} + Cz^{\pi+1+2n}$ &c. And putting $\pi+1 = p$, then $\dot{y} = pAz^\pi + \overline{p+n}\times Bz^{\pi+n} + \overline{p+2n}\times Cz^{\pi+2n}$ &c. put $m = \mu+1$, then

$$\begin{array}{r|llll}
-ez^{\pi}-fz^{\pi+n}\ \&c. & -ez^{\pi} & -fz^{\pi+n} & -gz^{\pi+2n} & \&c. \\
+a\dot{y} & +p\alpha A & +\overline{p+n}\times\alpha B & +\overline{p+2n}\times\alpha C & \&c. \\
+\beta z^{n}\dot{y} & & +p\beta A & +\overline{p+n}\times\beta B & \&c. \\
+\gamma z^{2n}\dot{y} & & & +\quad p\gamma A & \&c. \\
\&c. & & & & \\
+mn\beta yz^{n-1} & & +mn\beta A & +\quad mn\beta B & \&c. \\
+2mn\gamma yz^{2n-1} & & & +\quad 2mn\gamma A & \&c. \\
\&c. & & & &
\end{array}$$

$$\text{Hence } A=\frac{e}{p\alpha},\ B=\frac{f\left.\begin{matrix}-p\\-mn\end{matrix}\right\}\beta A}{\overline{p+n}\times\alpha},\ C=\frac{g\left.\begin{matrix}-p\\-2mn\end{matrix}\right\}\gamma A\left.\begin{matrix}-\overline{p+n}\\-mn\end{matrix}\right\}\beta B}{\overline{p+2n}\times\alpha}$$

$$D=\frac{h\left.\begin{matrix}-p\\-3mn\end{matrix}\right\}\delta A\left.\begin{matrix}-\overline{p+n}\\-2mn\end{matrix}\right\}\gamma B\left.\begin{matrix}-\overline{p+2n}\\-mn\end{matrix}\right\}\beta C}{\overline{p+3n}\times\alpha},\quad E=$$

$$\frac{k\left.\begin{matrix}-p\\-4mn\end{matrix}\right\}\varepsilon A\left.\begin{matrix}-\overline{p+n}\\-3mn\end{matrix}\right\}\delta B\left.\begin{matrix}-\overline{p+2n}\\-2mn\end{matrix}\right\}\gamma C\left.\begin{matrix}-\overline{p+3n}\\-mn\end{matrix}\right\}\beta D}{\overline{p+4n}\times\alpha},\ \&c.$$

$$\text{And } x=\overline{Az^{p}+Bz^{p+n}+Cz^{p+2n}+Dz^{p+3n}\ \&c.}\times\overline{a+\beta z^{n}+\gamma z^{2n}+\delta z^{3n}\ \&c.}^{\mu+1}$$

IV.

The *Fluent* of an *irrational Fluxion* may ſometimes alſo be found by *aſſuming* an *indetermin'd Series* as in the laſt Rule.

Ex. 36.

Suppoſe $\dot{z}=v^{2}x^{2}\dot{x}$, where $\dot{v}=\frac{a\dot{x}}{y}$, and $y=\sqrt{2ax-xx}$. I take $Av^{2}x^{3}$ for the firſt Term, and aſſume as many Terms of the inferiour Powers of v and x, or their Products, as I think will be ſufficient; for which no general Rule can be given. But you need take no more than the firſt Power of y, becauſe all the Powers above will be expreſs'd by the Powers of x, which are ſuppoſed to be already in the Equation. Thus I aſſume $z = Av^2x^3 + Bv^2 + \overline{Cx^2 + Dx + E} \times vy + Fx^3 + Gx^2 + Hx = \text{Flu}^t.\ v^2x^2\dot{x}$. Put this Equation into Fluxions making $\dot{x} = 1$, and writing every where $\frac{a}{y}$ for $\dot{v}$; then reduce it from Fractions, writing $2ax - xx$ for yy where it occurs. Then collect ſeverally all the homologous Terms (or thoſe of the ſame Powers of all or any of the Quantities x, v, y), thus

$$\begin{array}{lllll} +\,3Av^2x^2y & +\,2aAx^3v & +\,5aCx^2v & +\,3aDxv & +\,2aBv \\ -\,1 & -\,3C & -\,2D & -\,E & +\,aE \end{array}$$

$$\left.\begin{array}{lll} +\,3Fx^2y & +\,aDxy & +\,aEy \\ +\,aC & +\,2G & +\,H \end{array}\right\} = 0.$$

The reſpective Coefficients then being equated, there will be found $A = \frac{1}{3}$, $C = \frac{2}{9}a$, $D = \frac{5}{9}aa$, $E = \frac{5}{3}a^3$, $B = -\frac{5}{6}a^3$, $F = -\frac{2}{27}aa$, $G = -\frac{5}{18}a^3$, $H = -\frac{5}{3}a^4$. And thence $z = \frac{x^3v^2}{3} - \frac{5}{6}a^3v^2 + \overline{\frac{2}{9}ax^2 + \frac{5}{9}aax + \frac{5}{3}a^3} \times vy - \frac{2}{27}a^2x^3 - \frac{5}{18}a^3x^2 - \frac{5}{3}a^4x$.

Note, If any of the Quantities B, C, D *&c.* come out equal to nothing, ſtill the Series will be true, *provided* they don't deſtroy the Quantity A. But if A *vaniſh* by reaſon of ſome of the other Quantities being nothing; or if they involve ſome *impoſſible Equations*, then the Series is *not* true; and you muſt try again by aſſuming more Terms of the Powers or Products of x, v, y. But in many Caſes it cannot be done in finite Terms.

V.

In a *fluxionary Equation* where the variable Quantity is very *great*, and you would expreſs the Fluent by an *aſcending Series:* Or in any very much *compounded*

fluxionary

fluxionary Quantity whoſe Fluent is required; take a given Quantity extremely *near* equal to the variable Quantity; then inſtead of that variable Quantity *ſubſtitute* into the *Equation* the *Sum* of this given Quantity and a new variable Quantity, and likewiſe the Fluxion for the Fluxion; then find the *Fluent* in ſimple Terms, and this will be a *Part* of the whole *Fluent* required; and the Operation *repeated* as often as neceſſary will give the whole Fluent.

Ex. 37.

Let $\dot{z} = \dot{x}\sqrt{aa+xx}$; ſuppoſe r very near equal to x, and put $r + v = x$, and $aa+rr=ss$, then $\dot{x}=\dot{v}$, and $\dot{z} = \dot{v} \times \sqrt{aa+rr+2rv+vv} = \dot{v}\sqrt{ss + 2rv + vv}$

$$= \dot{v} \times : s + \frac{2rv+vv}{2s} - \frac{\overline{2rv+vv}^{\,2}}{8s^3} + \frac{\overline{2rv+vv}^{\,3}}{16s^5}$$

$$- \&c. = \dot{v} \times : s + \frac{rv}{s} + \frac{ss-rr}{2s^3}v^2 - \frac{ss-rr}{2s^5}rv^3$$

$$+ \&c. = s\dot{v} + \frac{rv\dot{v}}{s} + \frac{aav^2\dot{v}}{2s^3} - \frac{aarv^3\dot{v}}{2s^5} \&c.$$

Whence $z = sv + \dfrac{rv^2}{2s} + \dfrac{aav^3}{6s^3} - \dfrac{aarv^4}{8s^5}$ &c.

Now in this, ſubſtitute $-v$ for v, and ſubſtract the Reſult from the laſt Equation, and then $z = 2sv + \dfrac{a^2v^3}{3s^3} -$ &c. And this is the Part of the Fluent correſponding to the Difference of the Quantities $r+v$ and $r-v$, or to $2v$, that is, to theſe two different Values of x. Hence if v be taken extremely ſmall, and there be aſſumed ſucceſſively for r, the Numbers or Quantities b, c, d, e, f, &c. in Arithmetic Progreſſion, whoſe common Difference is $2v$; that is, ſo that $2v=b-c=c-d=d-e$, &c. till f (the laſt Value of r) be $= v$; then the Sum of all the Parts correſponding to each (collected by the foregoing Series) will be the whole Fluent z required. Or if you will, you may expreſs the Quantities *b*, *c*, *d*, *e*, &c. by $r, \overset{\prime}{r}, \overset{\prime\prime}{r}, \overset{\prime\prime\prime}{r}$, &c.

Ex. 38.

Suppoſe $\dot{z} = \frac{1}{2}\dot{x}\sqrt{\dfrac{pp+\overline{d-\frac{bx}{a}}^{2}}{2ax-xx}}$; let $r+v=x$. then $\dot{z}=\frac{1}{2}\dot{v}\sqrt{\dfrac{pp+dd-\frac{2dbr}{a}-\frac{2dbv}{a}+\frac{bb}{aa}\times\overline{r+v}^{2}}{2ar+2av-rr-2rv-vv}} =$ (putting $ss=pp+dd-\dfrac{2dbr}{a}+\dfrac{bbrr}{aa}$, $t=\dfrac{2bbr}{aa}-\dfrac{2db}{a}$, $qq=2ar-rr$, and $n=2a-2r$,) $\frac{1}{2}\dot{v}\sqrt{\dfrac{ss+tv+\frac{bb}{aa}vv}{qq+nv-vv}}$. Then dividing the Numerator by the Denominator, and putting $e=\dfrac{tqq-nss}{q^4}$, $f=\dfrac{bb}{aaqq}+\dfrac{ss}{q^4}+\dfrac{n^2s^2-ntq^2}{q^5}$, we have $\dot{z}=\frac{1}{2}\dot{v}\sqrt{\dfrac{ss}{qq}+ev+fv^2 \text{ \&c.}} = \frac{1}{2}\dot{v}\times:\dfrac{s}{q}+\dfrac{qe}{2s}v+\dfrac{4qfs^2-q^3e^2}{8s^3}v^2$ &c. Whence $z=\dfrac{sv}{2q}+\dfrac{qe}{8s}v^2+\dfrac{4fs^2-q^2e^2}{48s^3}qv^3$ &c. In which ſubſtituting $-v$ for v, and ſubtracting the Reſult from the Equation, we have $z=\dfrac{sv}{q}+\dfrac{4fs^2-q^2e^2}{24s^3}qv^3$ + &c. for that Part of the Fluent belonging to $2v$, or to the Difference of the Quantities $r+v$ and $r-v$.

Where it is *difficult* to get *many Terms* of the Series, as in the laſt Example, v muſt be taken ſo much the *ſmaller*, and the Operations *oftner* repeated before we can obtain the whole Fluent: And the working with

Numbers inſtead of known Letters, may ſometimes be preferable, when the Quantities are very complex.

Beſides the *general Rules* before delivered, there are ſome *particular Rules* which in ſome Caſes will find the Fluent in *finite* Terms. As

VI.

When one of the *variable* Quantities is *wanting* in the Equation: Then *aſſume* for its Fluxion the Product of the *other* Fluxion and a *new* variable Quantity, which ſubſtitute for the other; and you will get an Equation which put into Fluxions, will give the *Value* of the exterminated Fluxion; and then the Fluent will give the *aſſumed* Quantity; and from thence the *other* Quantity will be had.

Ex. 39.

Let $y\dot{y}^3\dot{x} = a\dot{x}^4 + 2a\dot{x}^2\dot{y}^2 + a\dot{y}^4$, where x is wanting. Aſſume $\frac{z\dot{y}}{a} = \dot{x}$, and expunging $\dot{x}$, $aazy = z^4 + 2a^2z^2 + a^4$, whence $y = \frac{z^3}{aa} + 2z + \frac{aa}{z}$. In Fluxions $\dot{y} = \frac{3z^2\dot{z}}{aa} + 2\dot{z} - \frac{aa\dot{z}}{zz}$, therefore $\frac{z\dot{y}}{a}$ or $\dot{x} = \frac{3z^3\dot{z}}{a^3} + \frac{2z\dot{z}}{a} - \frac{a\dot{z}}{z}$. Whence the Fluent is $x = \frac{3z^4}{4a^3} + \frac{z^2}{a} - a \times 2.302585$ Log. z. therefore y being known, z will be known (by the Equation $y = \frac{z^3}{aa} + 2z + \frac{aa}{z}$), and conſequently x by the laſt Equation.

VII.

Sometimes the *Fluent* may be had, by firſt putting the Equation into *Fluxions*, making ſome of the Fluxions *invariable*.

Ex. 40.

Let $\frac{a\dot{x}+y\dot{x}}{\dot{y}} = x + y - \frac{x\dot{y}}{\dot{x}}$. Make $\dot{y}$ conſtant, and put the Equation into Fluxions, then $\frac{\overline{a+y} \times \ddot{x}}{\dot{y}} + \dot{x} = \dot{x} + \dot{y} + \frac{x\dot{y}\ddot{x} - \dot{x}^2\dot{y}}{\dot{x}^2}$, $=$ and $\frac{\overline{a+y}\times\ddot{x}}{\dot{y}} = \frac{x\dot{y}\ddot{x}}{\dot{x}^2}$, whence $\overline{a+y}\times\dot{x}^2 = x\dot{y}^2$, and $\frac{\dot{x}}{\sqrt{x}} = \frac{\dot{y}}{\sqrt{a+y}}$, and the Fluent is $\sqrt{x}=\sqrt{a+y}$.

VIII.

Sometimes the *Fluent* may be found by *aſſuming* other *variable* Quantities to make up the Fluent, and finding their Values by Help of the *given* Equations and their Fluxions.

Ex. 41.

Suppoſe $y\dot{x} - x\dot{x} = a\dot{y}$. To find x.

Divide be $y - x$, and $\dot{x} = \frac{a\dot{y}}{y-x}$. Aſſume v, and ſuppoſe the Fluent $x=a\times 2.302585 \text{ Log.} \overline{y-x+v}$. then will $\dot{x} = \frac{a\dot{y} - a\dot{x} + a\dot{v}}{y-x+v}$ (ſee Prob. 2. Sect. II.) and by multiplying, $y\dot{x} - x\dot{x} + v\dot{x} = a\dot{y} - a\dot{x} + a\dot{v}$, from this ſubtract the given Equation $y\dot{x} - x\dot{x} = a\dot{y}$, and there remains $v\dot{x} = -a\dot{x} + a\dot{v}$, whence if $\dot{v} = 0$, then will $v = -a$; therefore $x = 2.302585a \times \text{Log.}\, \overline{y - x - a}$.

Ex. 42.

Suppoſe $a\dot{z} = z\dot{x} - x\dot{x}$. Aſſume $z = a + x + v$, then $\dot{z} = \dot{x} + \dot{v}$, the Values of z and $\dot{z}$ ſubſtituted in the given Equation give $a\dot{x} + a\dot{v} = a\dot{x} + x\dot{x} + v\dot{x} - x\dot{x}$, that is $a\dot{v} = v\dot{x}$, or $\dot{x} = \frac{a\dot{v}}{v} = \frac{ac\dot{v}}{cv}$, c being ſome given Quantity; whence $x = a \times 2.302585\, \text{Log.}\, \frac{v}{c}$, therefore $z = a + v + a \times 2.302585\, \text{Log.}\, \frac{v}{c}$. Or $z = a + x + c \times$ Number of the Logarithm $\frac{x}{2.30258a}$.

Ex. 43.

Suppoſe $\dot{z} = X^n x^m \dot{x}$, where X is the Hyperbolic Logarithm of x. Aſſume $z = \frac{X^n x^{m+1}}{m+1} + s$; this put into Fluxions there ariſes $\dot{z} = X^n x^m \dot{x} + \frac{nX^{n-1}\dot{X} \times x^{m+1}}{m+1} + \dot{s} = X^n x^m \dot{x}$, therefore $\dot{s} = \frac{-nX^{n-1}x^m\dot{x}}{m+1}$ (writing $\frac{\dot{x}}{x}$ for $\dot{X}$); Again aſſume $s = \frac{-nX^{n-1}x^{m+1}}{\overline{m+1}^2} + t$, this in Fluxions gives $\dot{t} = \frac{n \times \overline{n-1} \times X^{n-2}x^m\dot{x}}{\overline{m+1}^2}$. Then again $t = \frac{n \times \overline{n-1} \times X^{n-2}x^{m+1}}{\overline{m+1}^3} + u$. Whence $\dot{u} = \frac{-n \times \overline{n-1} \times \overline{n-2} \times X^{n-3}x^m\dot{x}}{\overline{m+1}^3}$, and $u =$

—

$$-\frac{n\times\overline{n-1}\times\overline{n-2}\times X^{n-3}x^{m+1}}{\overline{m+1}^{4}}+w,\quad\&c.$$

Whence $$z=\frac{X^{n}x^{m+1}}{m+1}-\frac{nX^{n-1}x^{m+1}}{\overline{m+1}^{2}}+\frac{n\times\overline{n-1}\times X^{n-2}x^{m+1}}{\overline{m+1}^{3}}-\frac{n\times\overline{n-1}\times\overline{n-2}\times X^{n-3}x^{m+1}}{\overline{m+1}^{4}}$$ + &c.

IX.

For compounded *fluxional Quantities*, aſſume an Equation with *indetermin'd Coefficients*, to comprehend the *Fluxion* given; wherefore the *Equation* muſt be ſuch that when put into Fluxions the *higheſt Power* may be the ſame as that of the Fluxion propoſed. Then comparing the homologous Terms, the *Coefficients* will he determin'd. And ſometimes the *Indices* may be found this Way.

Ex. 44.

To find the Fluent of $\overline{pyy+qay}\sqrt{yy+ay+aa}\times\dot{y}$.

Aſſume the imminential Equation $\overline{Ay+B}\times\overline{yy+ay+aa}^{\frac{3}{2}}$, in Fluxions

$$\left.\begin{array}{l}Ayy+Aay+Aaa\\ 3Ayy+3By\\ \qquad+\tfrac{3}{2}Aay+\tfrac{3}{2}Ba\end{array}\right\}\dot{y}\sqrt{yy+ay+aa}=$$

$$\overline{pyy+qay}\times y\sqrt{yy+ay+aa}.$$

Here $4A=p$, $Aa+3B+\frac{3}{2}Aa=qa$, $Aa^2+\frac{3}{2}Ba=0$.

Whence $A=\frac{1}{4}p$, $B=\frac{-ap}{6}$, and likewiſe $B=\frac{1}{3}qa-\frac{5}{24}ap$. Now if the Fluent be poſſible, theſe two Values of B muſt be equal, whence $8q=p$.

If the Fluent cannot be had this Way, aſſume the Form of it different. If $\overline{Ay^3+By^2+Cy+D}\sqrt{yy+ay+aa}$, were aſſumed, then $p=8q$ as before.

If $\overline{pyy+qa}\sqrt{yy+ay+aa}\times\dot{y}$ be propoſed. If $\overline{Ay+B}\times\overline{yy+ay+aa}|^{\frac{3}{2}}$ be aſſumed, the Fluent can only be found when $16q=-p$.

If $\overline{Ay^3+By^2+Cy+D}\sqrt{yy+ay+aa}$, the Fluent can only be had when $3p=80q$.

Ex. 45.

To find the Fluent of $\overline{ax+by}\times\dot{x}+\overline{cx+dy}\times\dot{y}=0$. Aſſume for the Fluent $\overline{x+\alpha y}|^{\pi}\times\overline{x+\beta y}|^{\tau}=A$ a given Quantity; and taking the Logarithms π L: $\overline{x+\alpha y}+\tau$ L: $\overline{x+\beta y}=$ L: A. In Fluxions $\frac{\pi\dot{x}+\alpha\pi\dot{y}}{\pi+\alpha y}+\frac{\tau\dot{x}+\tau\beta\dot{y}}{x+\beta y}=0$; reduced $\overline{\pi+\tau}\times x\dot{x}+\overline{\beta\tau+\alpha\tau}\times y\dot{x}+\overline{\alpha\pi+\beta\tau}\times x\dot{y}+\overline{\alpha\beta\pi+\alpha\beta\tau}\times y\dot{y}=0$; and comparing the Terms with thoſe of the given Equation, when reduced; you'll have 4 Equations $\pi+\tau=a$, $\beta\times\overline{\pi+\tau}=b$, $\overline{\alpha\pi+\beta\tau}=c$, $\alpha\beta\pi+\alpha\beta\tau=d$. Whence putting $m=\sqrt{\overline{b+c}|^{2}-4ad}$, $\alpha=\frac{b+c-m}{2a}$, $\beta=\frac{b+c+m}{2a}$, $\pi=\frac{ab-ac+am}{2m}$, $\tau=\frac{-ab+ac+am}{2m}$. Then the Fluent is $\overline{2ax+\overline{b+c-m}\times y}|^{b-c+m}\times\overline{2ax+\overline{b+c+m}\times y}|^{c-b+m}=B$ a given Quantity.

After the ſame Manner the Fluent of $\overline{ax^2+bxy+cyy}\times\dot{x}+\overline{dx^2+fxy+gyy}\times\dot{y}=0$, may be found; aſſuming the Fluent $\overline{x+\alpha y}|^{\pi}\times\overline{x+\beta y}|^{\tau}\times\overline{x+\gamma y}|^{\varsigma}=A$ a conſtant Quantity; and ſo of others conſiſting of more Terms.

Ex. 46.

To find the Fluent of $\frac{a\dot{x} + bx\dot{x}}{cx + xx}$:

Let the Fluent be ſuppoſed $d \times$ Hyp. Log. $\overline{cx^r + x^{r+1}}$; in Fluxions $d \times \frac{rcx^{r-1}\dot{x} + \overline{r+1} \times x^r\dot{x}}{cx^r + x^{r+1}}$ or $\frac{drc\dot{x} + d \times \overline{r+1} \times x\dot{x}}{cx + xx} = \frac{ax + bx\dot{x}}{cx + xx}$; then $drc = a$, and $d \times \overline{r+1} = b$. Whence $r = \frac{a}{bc - a}$, and $d = \frac{bc - a}{c}$. Therefore the Fluent required is $\frac{bc - a}{c} \times$ Hyp. Log. $cx^{\frac{a}{bc-a}} + x^{\frac{bc}{bc-a}}$.

Here follow ſome more Examples wherein the Rules before laid down are promiſcuouſly uſed.

Ex. 47.

To find the Fluent of $\dot{z}F^t.\,\dot{z}F:\dot{z}F:\dot{z}$ &c. $\times F:y\dot{z}$.

Let $A = F:y\dot{z}$

$\dot{B} = z\dot{A} = yz\dot{z}$

$\dot{C} = z\dot{B} = z^2\dot{A} = yz^2\dot{z}$

$\dot{D} = z\dot{C} = z^2\dot{B} = z^3\dot{A} = yz^3\dot{z}$

&c.

And $\alpha = F: A\dot{z}$, $\beta = F: \alpha\dot{z}$, $\gamma = F: \beta\dot{z}$, $\delta = F:\gamma\dot{z}$, &c. and let $b = F:z\dot{\alpha}$, $c = F:z\dot{\beta}$, $d = F:z\dot{\gamma}$, &c. then

1. $F:A\dot{z} = zA - B$, as is evident by putting the Equation into Fluxions; for the ſame Reaſon $F:\alpha\dot{z} = z\alpha - b$, $F:\beta\dot{z} = z\beta - c$, $F:\gamma\dot{z} = z\gamma - d$, &c.

2. Since $\alpha = zA - B = F:A\dot{z}$, and b or $F:z\dot{\alpha} = F:Az\dot{z}$. Suppoſe the $F: Az\dot{z} = \frac{1}{2}z^2A + s$, this in

Fluxions gives $\dot{s} = -\frac{z^2\dot{A}}{2} = -\frac{1}{2}\dot{C}$, therefore $s = -\frac{1}{2}C$. Whence $b = \frac{1}{2}z^2A - \frac{1}{2}C$. And $F: \alpha\dot{z} = (z\alpha - b =)\ z^2A - zB - \frac{1}{2}z^2A + \frac{1}{2}C = \frac{1}{2}z^2A - zB + \frac{1}{2}C = \beta$.

3. $F: \beta\dot{z} = z\beta - c$; but c or $F: z\dot{\beta} = F: \alpha z\dot{z} = F: \overline{zA - B} \times z\dot{z}$; ſuppoſe its Fluent $= \frac{1}{3}z^3A - \frac{1}{2}z^2B + s$, this in Fluxions gives $\dot{s} = -\frac{1}{3}z^3\dot{A} + \frac{1}{2}z^2\dot{B} = \frac{1}{6}\dot{D}$. Whence $s = \frac{1}{6}D$; and $c = \frac{1}{3}z^3A - \frac{1}{2}z^2B + \frac{1}{6}D$; and therefore $F: \beta\dot{z} = \frac{z^3A - 3z^2B + 3zC - D}{6} = \gamma$.

4. By the ſame Method $F: \gamma\dot{z} = \frac{z^4A - 4z^3B + 6z^2C - 4zD + E}{24}$. And ſo of others.

Whence

1 $Fl: y\dot{z} = A$.

2 $F: \dot{z}\,F: y\dot{z} = zA - B = \alpha = F: A\dot{z}$.

3 $F: \dot{z}\,F: \dot{z}\,F: y\dot{z} = \frac{z^2A - 2zB + C}{1 \times 2} = F: \alpha\dot{z} = \beta$.

4 $F: \dot{z}F: \dot{z}F: \dot{z}F: y\dot{z} = \frac{z^3A - 3z^2B + 3zC - D}{1 \times 2 \times 3} = F.\beta\dot{z} = \gamma$.

5 $F: \dot{z}\,F: \dot{z}\,F: \dot{z}\,F: \dot{z}\,F: y\dot{z} = \frac{z^4A - 4z^3B + 6z^2C - 4zD + E}{1 \times 2 \times 3 \times 4} = F: \gamma\dot{z} = \delta$.

&c.

After a like Manner the $F: z^{n-1} \times F: y\dot{z} = \frac{z^n}{n} \times F: y\dot{z} - \frac{F: yz^n\dot{z}}{n}$.

Ex. 48.

Suppoſing $A = F: y\dot{z}$, $\alpha = F: \dot{z}F: y\dot{z}$, $\beta = F: \dot{z}F: \dot{z}F: y\dot{z}$, $\gamma = F: \dot{z}F: \dot{z}F: \dot{z}F: y\dot{z}$, &c. as before. To find the Fluent of $z^n\dot{y}$.

Put F: $z^n\dot{y} = z^n y + s$; in Fluxions $z^n\dot{y} = z^n\dot{y} + nyz^{n-1}\dot{z} + \dot{s}$, therefore $\dot{s} = -nz^{n-1}y\dot{z} = -nz^{n-1}\dot{A}$, and $s = -nz^{n-1}A + t$, in Fluxions $\dot{t} = n \times \overline{n-1} \times Az^{n-2}\dot{z} = n \times \overline{n-1} \times z^{n-2}\dot{\alpha}$, and $t = n \times \overline{n-1} \times z^{n-2}\alpha + v$; in Fluxions $\dot{v} = -n \times \overline{n-1} \times \overline{n-2} \times z^{n-3}\alpha\dot{z} = -n \times \overline{n-1} \times \overline{n-2} \times z^{n-3}\dot{\beta}$, and $v = -n \times \overline{n-1} \times \overline{n-2} \times z^{n-3}\beta + w$, and $w = n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times z^{n-4}\gamma$, *&c.* Whence

F: $z^n\dot{y} = z^n y - nz^{n-1}A + n \times \overline{n-1} \times z^{n-2}\alpha - n \times \overline{n-1} \times \overline{n-2} \times z^{n-3}\beta + n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times z^{n-4}\gamma -$ *&c.*

Ex. 49.

Let $3x^3\dot{y} - 2axy\dot{y} = 3yx^2\dot{x} - ay^2\dot{x}$. Put $y = zx$, and $\dot{y} = z\dot{x} + x\dot{z}$, and expunging y, $\dot{y}$; $3x^2\dot{z} - 2azz\dot{x} - 2axz\dot{z} = -az^2\dot{x}$. Again put $x = zv$, and $\dot{x} = z\dot{v} + v\dot{z}$, which ſubſtituted in the laſt Equation, it becomes $3v^2\dot{z} - 3av\dot{z} = az\dot{v}$.

Let $v = \frac{aa}{s}$, and $\dot{v} = \frac{-aa\dot{s}}{ss}$, and the laſt Equation becomes $3a\dot{z} - 3s\dot{z} = -z\dot{s}$.

Let $s - a = t$, and $\dot{s} = \dot{t}$, and you'll get the Equation $3t\dot{z} = z\dot{t}$, or $3t\dot{z} - z\dot{t} = 0$, or $\frac{3\dot{z}}{z} - \frac{\dot{t}}{t} = 0$, and the Fluent z^3t^{-1} or $\frac{z^3}{t} = b = \frac{1}{a}$. Whence by Reſtitution, $y^3 + x^3 = axy$.

Ex. 50.

To find the Fluent of $az\dot{x} + \overline{b+rx} \times \dot{z} = \overline{e+fx} \times \dot{x}$; put $y = b + rx$, then the Equation is transform'd into $\frac{a}{r}z\dot{y} + y\dot{z} = \overline{e+fx}\times\dot{x}$; multiply by $y^{\frac{a}{r}-1} = \overline{b+rx}^{\frac{a}{r}-1}$, then will $\frac{a}{r}zy^{\frac{a}{r}-1}\dot{y} + y^{\frac{a}{r}}\dot{z} = \overline{e+fx}\times\dot{x}\times\overline{b+rx}^{\frac{a}{r}-1}$. And the Fluent is $zy^{\frac{a}{r}} = \frac{\overline{b+rx}^{\frac{a}{r}}}{a}\times\overline{e+fx} - \frac{f}{aa+ra}\times\overline{b+rx}^{\frac{a}{r}+1}$; divide by $y^{\frac{a}{r}} = \overline{b+rx}^{\frac{a}{r}}$, and you'll have $z = \frac{e+fx}{a} - \frac{f\times\overline{b+rx}}{aa+ar}$.

Ex. 51.

To find the Fluent of $\frac{a\dot{x}}{x} + \frac{b\dot{y}}{y} - \frac{c\dot{z}}{z} = 0$.

Multiply the Equation by $\frac{x^a y^b}{z^c}$, and we have $\frac{ax^{a-1}y^b\dot{x}}{z^c} + \frac{by^{b-1}x^a\dot{y}}{z^c} - \frac{cx^a y^b\dot{z}}{z^{c+1}}$; and the Fluent is $\frac{x^a y^b}{z^c} = \text{A}$ ſome given Quantity.

Ex. 52.

To find the Fluent of $\frac{\dot{p}}{p} = \frac{\dot{x}}{x} + \frac{\dot{y}}{y} - \frac{\dot{z}}{z}$.

The Fluent is Log. p = Log. x + Log. y — Log. z = Log. $\frac{xy}{z}$; and therefore $p = \frac{xy}{z}$.

Ex. 53.

To find the Fluent of $\overline{\alpha+\beta z^n}^{\mu} z^{\pi}\dot{z} = \dot{F}$.

Let $F = \dfrac{\overline{\alpha+\beta z^n}^{\mu+1}}{\overline{\mu+1}\times n\beta} z^{\pi+1-n} + s$; this in Fluxions gives $\dot{s} = -\dfrac{\overline{\pi+1-n}\times\overline{\alpha+\beta z^n}^{\mu+1}}{\overline{\mu+1}\times n\beta} z^{\pi-n}\dot{z}$, ſuppoſe $s = -\dfrac{\overline{\pi+1-n}\times\overline{\alpha+\beta z^n}^{\mu+2}}{\overline{\mu+1}\times\overline{\mu+2}\times n^2\beta^2} z^{\pi+1-2n} + t$, in Fluxions $\dot{t} = \dfrac{\overline{\pi+1-n}\times\overline{\pi+1-2n}\times\overline{\alpha+\beta z^n}^{\mu+2}}{\overline{\mu+1}\times\overline{\mu+2}\times n^2\beta^2} z^{\pi-2n}\dot{z}$, ſuppoſe $t = \dfrac{\overline{\pi+1-n}\times\overline{\pi+1-2n}\times\overline{\alpha+\beta z^n}^{\mu+3}}{\overline{\mu+1}\times\overline{\mu+2}\times\overline{\mu+3}\times n^3\beta^3} z^{\pi+1-3n} + v$, &c. Whence $F = \dfrac{\overline{\alpha+\beta z^n}^{\mu+1}}{\overline{\mu+1}\times n\beta} z^{\pi+1-n} + s + t + v$, &c.

That is Fl: $\overline{\alpha+\beta z^n}^{\mu} z^{\tau}\dot{z} = \dfrac{\overline{\alpha+\beta z^n}^{\mu+1}}{\overline{\mu+1}\times n\beta} z^{\pi+1-n} - \dfrac{\overline{\pi+1-n}\times\overline{\alpha+\beta z^n}}{\overline{\mu+2}\times n\beta z^n} A + \dfrac{\overline{\pi+1-2n}\times\overline{\alpha+\beta z^n}}{\overline{\mu+3}\times n\beta z^n} B - \dfrac{\overline{\pi+1-3n}\times\overline{\alpha+\beta z^n}}{\overline{\mu+3}\times n\beta z^n} C +$ &c.

Otherwiſe.

Suppoſe $F = \dfrac{\overline{\alpha+\beta z^n}^{\mu} z^{\pi+1}}{\pi+1} + s$, then $\dot{s} = -\dfrac{\mu n\beta\times\overline{\alpha+\beta z^n}^{\mu-1} z^{\tau+n}\dot{z}}{\pi+1}$:

Let $s = \frac{-\mu n\beta \times \overline{\alpha+\beta z^n}|^{\mu-1} z^{\pi+n-1}}{\overline{\pi+1} \times \overline{\pi+1+n}} + t$; whence $\dot{t} = \frac{\mu \times \overline{\mu-1} \times n^2\beta^2 \times \overline{\alpha+\beta z^n}|^{\mu-2} z^{\pi+2n}\dot{z}}{\overline{\pi+1} \times \overline{\pi+1+n}}$, and let $t = \frac{\mu \times \overline{\mu-1} \times n^2\beta^2 \times \overline{\alpha+\beta z^n}|^{\mu-2} z^{\pi+2n+1}}{\overline{\pi+1} \times \overline{\pi+1+n} \times \overline{\pi+1+2n}} + v$, &c.

Whence F: $\overline{\alpha+\beta z^n}|^{\mu} z^{\pi}\dot{z} = \frac{\overline{\alpha+\beta z^n}|^{\mu} z^{\pi+1}}{\pi+1} - \frac{\mu n\beta z^n}{\overline{\pi+1+n} \times \overline{\alpha+\beta z^n}} A + \frac{\overline{\mu-1} \times n\beta z^n}{\overline{\pi+1+2n} \times \overline{\alpha+\beta z^n}} B - \frac{\overline{\mu-2} \times n\beta z^n}{\overline{\pi+1+3n} \times \overline{\alpha+\beta z^n}} C +$ *&c.*

Ex. 54.

To find the Fluent of $\overline{\alpha+\beta z^n+\gamma z^{2n}+\delta z^{3n}\ \&c.}|^{\mu} \times z^{\pi}\dot{z}$, having the Fluent of $\overline{\alpha+\beta z^n}|^{\mu} z^{\pi}\dot{z}$ given.

Suppoſe $\overline{\alpha+\beta z^n+\gamma z^{2n}+\delta z^{3n}\ \&c.}|^{\mu} = \overline{\alpha+\beta z^n}|^{\mu} \times \overline{1+0+Cz^{2n}+Dz^{3n}\&c.}$ then $\overline{\alpha+\beta z^n+\gamma z^{2n}+\delta z^{3n}\&c.} = \overline{\alpha+\beta z^n} \times \overline{1+0+Cz^{2n}+\&c.}|^{\frac{1}{\mu}}$; but $\overline{1+0+Cz^{2n}+\&c.}|^{\frac{1}{\mu}} = 1+0+\frac{1}{\mu}Cz^{2n}+\frac{1}{\mu}Dz^{3n}$ $\left.\begin{array}{l} +\frac{1}{\mu}\times\frac{\frac{1}{\mu}-1}{2}CC \\ \quad +\frac{1}{\mu}E \end{array}\right\} z^{4n} +$ *&c.* Multiply this Series into $\overline{\alpha+\beta z^n}$, and we ſhall have $\alpha+\beta z^n+\gamma z^{2n}+\delta z^{3n}$ *&c.* $=$

α

$$\begin{array}{llll} \alpha + 0 & + \frac{\alpha}{\mu}Cz^{2n} & + \frac{\alpha}{\mu}Dz^{3n} & \\ + \beta z^{n} + 0 & & + \frac{\beta}{\mu}C & \end{array}$$ &c. And equating the Coefficients of the ſeveral Powers, then $\alpha = \alpha$, $\beta = \beta$, $\gamma = \frac{\alpha}{\mu}C$, $\delta = \frac{\alpha}{\mu}D + \frac{\beta}{\mu}C$, &c. Whence $C = \frac{\mu\gamma}{\alpha}$, $D = \frac{\mu\delta}{\alpha} - \frac{\beta\mu\gamma}{\alpha\alpha}$, $E = \frac{\mu\varepsilon}{\alpha} - \frac{1-\mu}{2\mu}CC - \frac{\beta}{\alpha}D$, &c. Therefore Fl: $\overline{\alpha + \beta z^{n} + \gamma z^{2n} + \delta z^{3n} \&c.}^{\mu} \times z^{\pi}\dot{z} = $ F: $\overline{\alpha + \beta z^{n}}|^{\mu} z^{\pi}\dot{z} \times$ into $1 + \frac{\mu\gamma}{\alpha}z^{2n} + \frac{\alpha\delta - \beta\gamma}{\alpha\alpha}\mu z^{3n} + Ez^{4n}$ &c.

Therefore if the Fluent of $\overline{\alpha + \beta z^{n}}|^{\mu} z^{\pi}\dot{z}$ be known, the Fluent of this compound Fluxion will be known, by Form 11th of the Table, and this is uſeful when the Quantities γz^{2n}, δz^{3n}, &c. are very ſmall.

Likewiſe the whole Fluent when $\alpha + \beta z^{n}$ becomes $= 0$, will be found from this Rule, by Form 17th of the Table.

Ex. 55.

To find the Fluent of $z^{n}x^{r}y^{m}\dot{y}$.

Let $x^{r}y^{m}\dot{y} = \dot{A}$, $A\dot{z} = \dot{B}$, $B\dot{z} = \dot{C}$, &c. then the Fluxion given is $z^{n}\dot{A}$; then by Rule VIII. put F: $z^{n}\dot{A} = z^{n}A + s$; in Fluxions $z^{n}\dot{A} = z^{n}\dot{A} + nAz^{n-1}\dot{z} + \dot{s}$, or $\dot{s} = -nz^{n-1}A\dot{z} = -nz^{n-1}\dot{B}$; aſſume $s = -nz^{n-1}B + t$, then $\dot{t} = n \times \overline{n-1} \times z^{n-2}B\dot{z} = n \times \overline{n-1}\, z^{n-2}\dot{C}$: Let $t = n \times \overline{n-1} \times z^{n-2}C + u$; then $\dot{u} = -n \times \overline{n-1} \times \overline{n-2} \times z^{n-3}C\dot{z} = -n \times \overline{n-1} \times \overline{n-2} \times z^{n-3}\dot{D}$. Then aſſume $u = -n \times \overline{n-1} \times \overline{n-2} \times z^{n-3}D + w$, &c. Then at laſt F: $z^{n}x^{r}y^{m}\dot{y}$ or F: $z^{n}\dot{A} = z^{n}A - nz^{n-1}B +$

$n\times\overline{n-1}\times z^{n-2}C - n\times\overline{n-1}\times\overline{n-2}\times z^{n-3}D + n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times z^{n-4}E -$ *&c.*

Then by the very ſame Rule that theſe are had, you will get the Fluents of $\dot{A}$ or $x^r y^m \dot{y}$, and $\dot{B}$ or $A\dot{z}$, of $\dot{C}$ or $B\dot{z}$, *&c.* in any particular Caſe; expunging by Degrees $\dot{x}$ $\dot{y}$ or $\dot{z}$ as there is Occaſion, by Help of the given Equations. The above Series is the ſame as is found, Example 48.

Suppoſe the given Fluxion is $zx^2y^3\dot{y}$, and $rr - xx = yy$, and $-x\dot{x} = y\dot{y}$, alſo $x\dot{z} = -r\dot{y}$ and $y\dot{z} = r\dot{x}$; r, y, x being the Radius, Sine and Coſine in the Circle. Then we ſhall have $F: zx^2y^3\dot{y} = zA - B$, where $A = F: x^2y^3\dot{y}$, and $B = F: A\dot{z}$; here $A = F: \overline{rr - yy}\times y^3\dot{y} = \frac{r^2y^4}{4} - \frac{y^6}{6}$, and $B = F: \overline{\frac{1}{4}rr - \frac{1}{6}yy}\times y^4\dot{z} = \overline{\frac{1}{4}rr - \frac{1}{6}yy}\times y^3\times r\dot{x} = \overline{\frac{1}{4}rr - \frac{1}{6}yy}\times\frac{-ry^4\dot{y}}{\sqrt{rr-yy}}$, whoſe Fluent is found from the Table.

And thus I have explain'd, as clearly as I could, the moſt *general Methods* of ſolving this *difficult Problem.* The Reader perhaps may think me too prolix on this Head: But it muſt be conſider'd that this is a Problem of the *greateſt Extent* and *Uſe* in the whole Practice of Fluxions; nay it contains almoſt the whole Science; and we cannot be too particular in treating on that which is the *Foundation* of the greateſt Part of the *Practice.* But after all, we ſhall find it exceeding difficult in many Caſes to find the Fluents of Quantities, by any Methods hitherto known. And it is much to be wiſh'd that we had ſome eaſier Methods of finding Fluents, eſpecially of compound fluxionary Quantities, without the tedious Labour of reducing them to infinite Series, which in many Caſes converge ſo ſlow, and are ſo much compounded as to be in a manner uſeleſs. The following Propoſition is deſign'd to remedy this Difficulty in ſome particular Caſes.

PROP. XI.

To find the Fluent of a given Fluxion by the Table.

1. The following Table comprehends all Sorts of *Fluxions* and their correſpondent *Fluents*; not only ſuch as can be exactly had in *finite Terms*, and thoſe depending on the *Quadrature* of the *Conic Sections*, but alſo thoſe that can only be had by *infinite Series*.

2. Theſe Forms are all number'd in the firſt Column; the ſecond Column contains the Fluxions, and the third gives the Fluent thereof. The 21ſt and all the following Forms relate to the *Tranſmutation* of Fluxions; here the Fluxions in the ſecond and third Columns are equal, the ſecond being transform'd into the third, and the Fluxion in the third Column always belongs to ſome of the foregoing Forms.

3. Here z, v, y expreſs *variable Quantities*, and all the reſt are *given* ones, which may repreſent any Quantities whatſoever affirmative or negative. But in the 6th, 8th, and 10th Forms, the Nature of them requires *negative* Quantities, and therefore they are witten *negative*.

4. In the ſecond Column are ſet down all the *neceſſary Conditions* relating to the *Signs*, *Indices*, &c. in each Form: likewiſe in what Caſes the *Form* will *fail*, and when the Series will *terminate*. But the Fluents in Form 11, 12, 13, and 14 are deſign'd always to terminate, and are derived from the foregoing ten Forms, which therefore may be called *original Forms*.

5. In theſe original Forms, though the *firſt* of the Fluents there given may in moſt Caſes be ſufficient, yet there are ſeveral *Varieties* both of *numerical* and *geometrical Fluents*; ſo that the moſt ſimple and elegant may always be choſen to ſuit any particular Caſe. It is ſufficient to premiſe this concerning the Nature of theſe Forms.

A TABLE

Of FLUXIONS and FLUENTS.

L = 2.302585092994045684 *&c.*
N = ,017453292519943295 *&c.*

Forms.	*Fluxions.*	*Fluents.*
1	$z^{n-1}\dot{z}$ *This Form fails when* $n = 0$.	$\frac{z^n}{n}$.
2	$z^{-1}\dot{z} = \dot{\varphi}$	$\varphi = \mathrm{L} \times$ Log: z. Or $\mathrm{L} \times$ Log: Az. Or $\varphi = \frac{Area}{\mathrm{RR}}$ of the Hyperbola between the Aſſyptotes, whoſe inſcribed Parallelogram is any Space RR, and Abſciſa (taken in the Aſſymptote) Rz or z.
3	$\overline{\alpha + \beta z^n}^{\mu} z^{n-1}\dot{z}$ *This Form fails when* $\mu = -1$.	$\frac{\overline{\alpha + \beta z^n}^{\mu+1}}{\overline{\mu+1} \times n\beta}$.
4	$\frac{z^{n-1}\dot{z}}{\alpha + \beta z^n} = \dot{\varphi}$	$\varphi = \frac{\mathrm{L}}{n\beta} \times$ Log. of $\overline{\alpha + \beta z^n}$. $\varphi = \frac{Area}{n\beta \mathrm{RR}}$ of an Hyperbola between the Aſſymptotes, whoſe inſcribed Parallelogram is RR, Abſciſſa $\frac{\alpha}{\beta} + z^n$.

	Fluxions.	*Fluents.*
5	$\frac{z^{\frac{1}{2}n-1}\dot{z}}{\alpha+\beta z^n}=\dot{\varphi}$ *Here α and β are affirmative.*	$\varphi=\frac{2N}{n\sqrt{\alpha\beta}}\times$ *Degrees* in that Arch of a Circle whoſe Radius is 1, and natural Tangent $\sqrt{\frac{\beta z^n}{\alpha}}$. $\varphi=\frac{\mathit{Arch}}{\frac{1}{2}nR\sqrt{\alpha\beta}}$ of a Circle whoſe Radius is any Line R, and Tangent $R\sqrt{\frac{\beta z^n}{\alpha}}$. $\varphi=\frac{4\,\mathit{Sectors}}{nRR\sqrt{\alpha\beta}}$ of the Circle whoſe Radius is R, and Tangent $R\sqrt{\frac{\beta z^n}{\alpha}}$.
6	$\frac{z^{\frac{1}{2}n-1}\dot{z}}{\alpha-\beta z^n}=\dot{\varphi}$ *Here α is affirmative, and $-\beta$ negative.*	$\varphi=\frac{L}{n\sqrt{\alpha\beta}}\times$ vulg. Log. of $\frac{\sqrt{\alpha}+\sqrt{\beta z^n}}{\sqrt{\alpha}-\sqrt{\beta z^n}}$. $\varphi=\frac{L}{n\sqrt{\alpha\beta}}\times$ Log. of $\frac{\sqrt{\beta z^n}+\sqrt{\alpha}}{\sqrt{\beta z^n}-\sqrt{\alpha}}$. $\varphi=\frac{L}{n\sqrt{\alpha\beta}}\times$ Log. of $\frac{\alpha+\beta z^n+2\sqrt{\alpha\beta z^n}}{\alpha-\beta z^n}$. $\varphi=\frac{L}{n\sqrt{\alpha\beta}}\times$ Log. of $\frac{\alpha-\beta z^n}{\alpha+\beta z^n-2\sqrt{\alpha\beta z^n}}$. $\varphi=\frac{\mathit{Area}}{nRR\sqrt{\alpha\beta}}$ of the Hyperbola between the Aſſymptotes, whoſe inſcribed Parallelogram is RR, and Abſciſſas (terminating this Area) $\sqrt{\alpha}+\sqrt{\beta z^n}$ and $\sqrt{\alpha}\backsim\sqrt{\beta z^n}$. $\varphi=\frac{4\,\mathit{Sectors}}{nRR\sqrt{\alpha\beta}}$ of a right angled Hyperbola, whoſe ſemitranſverſe is R, and Tangent at the Vertex $R\sqrt{\frac{\beta z^n}{\alpha}}$.

Forms.	*Fluxions.*
7	$\frac{z^{\frac{\delta}{\lambda}\eta-1}\dot{z}}{\alpha+\beta z^{\eta}}$ *Here* α *and* β *are affirmative.* $\frac{\delta}{\lambda}$ *is an affirmative proper Fraction.*

Fluents.

$$\frac{1}{\eta\alpha^{\frac{\lambda-\delta}{\lambda}}\beta^{\frac{\delta}{\lambda}}}\times \text{ into } AL\times\text{Log.}\sqrt{1-2sx+xx}+BN\times P$$
$$+\,CL\times\text{Log.}\sqrt{1-2tx+xx}+DN\times Q$$
$$+\,EL\times\text{Log.}\sqrt{1-2ux+xx}+FN\times R \text{ \&c.}$$

K = an *Arch* of $\frac{180 \text{ Deg.}}{\lambda}$. $x = \overline{\frac{\beta z^{n}}{\alpha}}\Big|^{\frac{1}{\lambda}}$.

$\left.\begin{array}{l} a,\ b,\ c, \text{ \&c.} = \text{Sines} \\ s,\ t,\ u, \text{ \&c.} = \text{Cofines} \end{array}\right\}$ of 1K, 3K, 5K &c. to 180°. Radius = 1.

P, Q, R, &c. = *Degrees* of Arches whoſe Sines are $\frac{ax}{\sqrt{1-2sx+xx}}$, $\frac{bx}{\sqrt{1-2tx+xx}}$, $\frac{cx}{\sqrt{1-2ux+xx}}$, &c.

CASES.

1. If $\frac{\delta}{\lambda}=\frac{1}{3}$, then $A=(-2s=)-1$. $B=2a$. $C=1$. $D=0=E=F$ &c.

2. If $\frac{\delta}{\lambda}=\frac{2}{3}$, $A=(2s=)1$. $B=2a$. $C=-1$. D, E, F &c. $=0$.

$$\left.\begin{array}{l} 3.\ \text{If } \frac{\delta}{\lambda}=\frac{1}{4},\ A=-2s.\ B=2a.\ C=-2t.\ D=2b. \\ 4.\ \text{If } \frac{\delta}{\lambda}=\frac{3}{4},\ A=2s.\ B=2a.\ C=2t.\ D=2b. \end{array}\right\} E, F \text{ \&c.} = 0.$$

5. If $\frac{\delta}{\lambda}=\frac{1}{5}$, $A=-2s$. $B=2a$. $C=-2t$. $D=2b$.
   $E=1$. F &c. $=0$.

6. If $\frac{\delta}{\lambda}=\frac{2}{5}$, $A=2t$. $B=2b$. $C=2s$. $D=-2a$.
   $E=-1$. F &c. $=0$.

7. If $\frac{\delta}{\lambda}=\frac{1}{6}$, $A=-2s$. $B=2a$. $C=0$. $D=2b$.
   $E=-2u$. $F=2c$. &c. $=0$.

&c.

Forms.	*Fluxions.*
8	$\frac{z^{\frac{\delta}{\lambda}\eta-1}\dot{z}}{\alpha-\beta z^{\eta}}$ *Here* α *is affirmative, and* $-\beta$ *negative.* $\frac{\delta}{\lambda}$ *is an affirmative proper Fraction.*

Fluents.

$$\frac{1}{\eta\beta^{\frac{\delta}{\lambda}}\alpha^{\frac{\lambda-\delta}{\lambda}}} \times \text{ into } AL \times \text{Log.}\ \overline{1-x}. + BL \times \text{Log.}\ \sqrt{1-2sx+xx} + CN \times P + DL \times \text{Log.}\ \sqrt{1-2tx+xx} + EN \times Q + FL \times \text{Log.}\ \sqrt{1-2ux+xx} + GN \times R\ \&c.$$

$K =$ an *Arch* of $\frac{180\ \text{Deg.}}{\lambda}$. $x = \overline{\frac{\beta z^n}{\alpha}}\Big|^{\frac{1}{\lambda}}$.

$$\left.\begin{array}{l} a,\ b,\ c,\ \&c. = \text{Sines} \\ s,\ t,\ u,\ \&c. = \text{Coſines} \end{array}\right\} \begin{array}{l} \text{of } 2K,\ 4K,\ 6K\ \&c. \text{ to } 180^\circ. \\ \text{Radius} = 1. \end{array}$$

$P, Q, R,$ &c. = *Degrees* of Arches whoſe Sines are $\frac{ax}{\sqrt{1-2sx+xx}}$, $\frac{bx}{\sqrt{1-2tx+xx}}$, $\frac{cx}{\sqrt{1-2ux+xx}}$, &c.

CASES.

1. If $\frac{\delta}{\lambda} = \frac{1}{3}$, then $A = -1$, $B = -2s$, $C = 2a$.

2. If $\frac{\delta}{\lambda} = \frac{2}{3}$, $A = -1$, $B = -2s$, $C = -2a$, D, E &c. $= 0$.

3. If $\frac{\delta}{\lambda} = \frac{1}{4}$, $A = -1$, $B = (-2s =)\ 0$, $C = 2a$, $D = 1$.

4. If $\frac{\delta}{\lambda} = \frac{3}{4}$, $A = -1$, $B = 0$, $C = -2a$, $D = 1$, E, F &c. $= 0$.

5. If $\frac{\delta}{\lambda} = \frac{1}{5}$, $A = -1$, $B = -2s$, $C = 2a$, $D = -2t$, $E = 2b$.

6. If $\frac{\delta}{\lambda} = \frac{2}{5}$, $A = -1$, $B = -2t$, $C = 2b$, $D = -2s$, $E = -2a$, F &c. $= 0$.

7. If $\frac{\delta}{\lambda} = \frac{1}{6}$, $A = -1$, $B = -2s$, $C = 2a$, $D = -2t$, $E = 2b$, $F = 1$, G, &c. $= 0$.

&c.

Forms.	*Fluxions.*
9	$\dfrac{z^{\frac{1}{2}\eta-1}\dot{z}}{\sqrt{\alpha+\beta z^{\eta}}}=\dot{\phi}$ *Here β is affirmative.*

Fluents.

$$\phi = \frac{2L}{n\sqrt{\beta}} \times \text{Log. of } \overline{\sqrt{\beta z^n} + \sqrt{\alpha + \beta z^n}}\,.$$

$$\phi = \frac{L}{n\sqrt{\beta}} \times \text{Log. of } \overline{\alpha + 2\beta z^n + 2\sqrt{\alpha + \beta z^n \times \beta z^n}}\,.$$

$$\phi = \frac{-2L}{n\sqrt{\beta}} \times \text{Log. of } \overline{\sqrt{\alpha + \beta z^n} \backsim \sqrt{\beta z^n}}\,.$$

$$\phi = \frac{-L}{n\sqrt{\beta}} \times \text{Log. of } \overline{\alpha + 2\beta z^n - 2\sqrt{\alpha\beta z^n + \beta^2 z^{2n}}}\,.$$

$\phi = \dfrac{4\,\textit{Sectors}}{n RR\sqrt{\beta}}$ of the right angled Hyperbola, whoſe Semitranſverſe is R, and Ordinate $R\sqrt{\dfrac{\beta z^n}{\alpha}}$.

$\phi = \dfrac{2}{n RR\sqrt{\beta}} \times$ *Sector* of the right Hyperbola, whoſe Semitranſverſe is R, and Ordinate $\dfrac{2R}{\alpha}\sqrt{\alpha + \beta z^n \times \beta z^n}$.

$\phi = \dfrac{2}{n RR\sqrt{\beta}} \times$ *Sector* of the right Hyperbola, whoſe Semitranſverſe is R, and Ordinate $\dfrac{2R}{-\alpha}\sqrt{\alpha + \beta z^n \times \beta z^n}$, when α ſtands for a negative Quantity.

$\phi = \dfrac{4\,\textit{Sectors}}{n RR\sqrt{\beta}}$ of the right Hyperbola, whoſe Semitranſverſe is R, and Ordinate $\dfrac{R}{\sqrt{-\alpha}} \times \sqrt{\alpha + \beta z^n}$, when α ſtands for a negative Quantity.

Forms.	*Fluxions.*
10	$\frac{z^{\frac{1}{2}n-1}\dot{z}}{\sqrt{\alpha-\beta z^{n}}}=\dot{\phi}$
	Here α *is affirmative, and* $-\beta$ *is negative.*

Fluents.

$\phi = \frac{2N}{n\sqrt{\beta}} \times$ *Degrees* in the Arch of a Circle whoſe Radius is 1, and natural Sine $\sqrt{\frac{\beta z^n}{\alpha}}$.

$\phi = \frac{N}{n\sqrt{\beta}} \times$ *Degrees* in the Arch whoſe Radius is 1, and Sine $\frac{2}{\alpha}\sqrt{\alpha - \beta z^n \times \beta z^n}$.

$\phi = \frac{2}{nR\sqrt{\beta}} \times$ *Arch* whoſe Radius is R, and Sine $R\sqrt{\frac{\beta z^n}{\alpha}}$.

$\phi = \frac{Arch}{nR\sqrt{\beta}}$, whoſe Radius is R, and Sine $\frac{2R}{\alpha}\sqrt{\alpha - \beta z^n \times \beta z^n}$, or verſed Sine $\frac{2R}{\alpha}\beta z^n$.

$\phi = \frac{4}{nRR\sqrt{\beta}} \times$ *Sector* of a Circle, whoſe Radius is any Line R, and Sine $R\sqrt{\frac{\beta z^n}{\alpha}}$.

$\phi = \frac{2\ Sectors}{nRR\sqrt{\beta}}$ of the Circle whoſe Radius is R, and Sine $\frac{2R}{\alpha}\sqrt{\alpha\beta z^n - \beta^2 z^{2n}}$.

Forms.	*Fluxions.*
11	$\overline{\alpha + \beta z^{n}}\vert^{\mu} z^{\pi+\lambda n}\dot{z}$ *Here* λ *is any affirmative whole Number.* N. B: *This Form fails when* $\frac{\pi+1}{n}+\mu$ *is any negative whole Number from* —1 *to* —λ *incluſive.*
12	$\overline{\alpha + \beta z^{n}}\vert^{\mu} z^{\pi-\lambda n}\dot{z}$ *Here* λ *is any affirmative whole Number.* N. B. *This Form fails when* $\frac{\pi+1}{n}$ *is any affirmative whole Number from* 1 *to* λ *incluſively.*

Fluents.

$$\frac{\overline{-\alpha}|^{\lambda}\times\overline{\varepsilon-\eta}\times\overline{\varepsilon-2\eta}\times\overline{\varepsilon-3\eta}\times\&c.\ \ldots\ \text{till}\ \overline{\varepsilon-\lambda\eta}}{\beta^{\lambda}\times\overline{\delta-\eta}\times\overline{\delta-2\eta}\times\overline{\delta-3\eta}\times\&c.\ \ldots\ \text{till}\ \overline{\delta-\lambda\eta}.}\,\Phi$$

$$+\frac{z^{\varepsilon-\eta}\gamma}{\overline{\delta-\eta}\times\beta}-\frac{\overline{\varepsilon-\eta}\times\alpha}{\overline{\delta-2\eta}\times\beta z^{\eta}}A-\frac{\overline{\varepsilon-2\eta}\times\alpha}{\overline{\delta-3\eta}\times\beta z^{\eta}}B$$

$$-\frac{\overline{\varepsilon-3\eta}\times\alpha}{\overline{\delta-4\eta}\times\beta z^{\eta}}C-\&c.\ \text{till}\ -\frac{\overline{\varepsilon+\eta-\lambda\eta}\times\alpha}{\overline{\delta-\lambda\eta}\times\beta z^{\eta}}F.$$

$\varphi=$ Fluent of $\overline{\alpha+\beta z^{\eta}}|^{\mu}z^{\pi}\dot{z}$.

$\gamma=\overline{\alpha+\beta z^{\eta}}|^{\mu+1}$.

$\varepsilon=\pi+1+\lambda\eta$.

$\delta=\varepsilon+\eta+\mu\eta$.

A, B, C *&c.* each preceding Term with its Sign.

$$\frac{\beta^{\lambda}\times\overline{\delta+\eta}\times\overline{\delta+2\eta}\times\overline{\delta+3\eta}\times\&c.\ \ldots\ \text{till}\ \overline{\delta+\lambda\eta}}{\overline{-\alpha}|^{\lambda}\times\overline{\varepsilon+\eta}\times\overline{\varepsilon+2\eta}\times\overline{\varepsilon+3\eta}\times\&c.\ \ldots\ \text{till}\ \overline{\varepsilon+\lambda\eta}}\,\Phi$$

$$+\frac{z^{\varepsilon+\eta}\gamma}{\overline{\varepsilon+\eta}\times\alpha}-\frac{\overline{\delta+\eta}\times\beta z^{\eta}}{\overline{\varepsilon+2\eta}\times\alpha}A-\frac{\overline{\delta+2\eta}\times\beta z^{\eta}}{\overline{\varepsilon+3\eta}\times\alpha}B$$

$$-\frac{\overline{\delta+3\eta}\times\beta z^{\eta}}{\overline{\varepsilon+4\eta}\times\alpha}C-\&c.\ \text{till}\ -\frac{\overline{\delta-\eta+\lambda\eta}\times\beta z^{\eta}}{\overline{\varepsilon+\lambda\eta}\times\alpha}F.$$

$\varphi=$ Fluent of $\overline{\alpha+\beta z^{\eta}}|^{\mu}z^{\pi}\dot{z}$.

$\gamma=\overline{\alpha+\beta z^{\eta}}|^{\mu+1}$.

$\varepsilon=\pi+1-\lambda\eta-\eta$.

$\delta=\varepsilon+\mu\eta+\eta$.

A, B, C *&c.* each preceding Term with its Sign.

Forms.	*Fluxions.*
13	$\overline{\alpha+\beta z^{\eta}}^{\mu+\tau} z^{\pi}\dot{z}$ *Here* τ *is any affirmative whole Number.* N. B. *This Form fails when* $\frac{\pi+1}{\eta}+\mu$ *is any negative whole Number from* -1 to $-\tau$ *incluſively.*
14	$\overline{\alpha+\beta z^{\eta}}^{\mu-\tau} z^{\pi}\dot{z}$ *Here* τ *is any affirmative whole Number.* N. B. *This Form will fail when* $\mu+1$ *is any affirmative whole Number from* 1 *to* τ *incluſive.*

Fluents.

$$\frac{\eta^{\tau}\alpha^{\tau}\times\overline{\mu+1}\times\overline{\mu+2}\times\overline{\mu+3}\times\text{\&c.}\ \ldots\ldots\ \text{till }\overline{\mu+\tau}}{\overline{\delta+\eta}\times\overline{\delta+2\eta}\times\overline{\delta+3\eta}\times\text{\&c.}\ \ldots\ldots\ \text{till }\overline{\delta+\tau\eta}}\phi$$

$$+\ \frac{z^{\pi+1}\gamma^{\mu+\tau}}{\overline{\delta+\tau\eta}}+\frac{\overline{\mu+\tau}\times\eta\alpha}{\overline{\delta+\tau\eta-\eta}\times\gamma}\mathrm{A}+\frac{\overline{\mu+\tau-1}\times\eta\alpha}{\overline{\delta+\tau\eta-2\eta}\times\gamma}\mathrm{B}$$

$$+\ \frac{\overline{\mu+\tau-2}\times\eta\alpha}{\overline{\delta+\tau\eta-3\eta}\times\gamma}\mathrm{C}+\text{\&c. till }\frac{\overline{\mu+2}\times\eta\alpha}{\overline{\delta+\eta}\times\gamma}\mathrm{F}.$$

$\phi =$ Fluent of $\overline{\alpha+\beta z^{\eta}}^{\,\mu} z^{\pi}\dot{z}$.

$\gamma = \alpha+\beta z^{\eta}$.

$\delta = \pi + 1 + \mu\eta$.

A, B, C *&c.* each preceding Term with its Sign.

$$\frac{\delta\times\overline{\delta-\eta}\times\overline{\delta-2\eta}\times\overline{\delta-3\eta}\times\text{\&c.}\ \ldots\ \text{till }\overline{\delta+\eta}\,\overline{-\tau\eta}}{\eta^{\tau}\alpha^{\tau}\mu\times\overline{\mu-1}\times\overline{\mu-2}\times\overline{\mu-3}\times\text{\&c.}\ \ldots\ \text{till }\overline{\mu+1-\tau}}\phi$$

$$-\ \frac{z^{\pi+1}\gamma^{\mu-\tau+1}}{\overline{\mu-\tau+1}\times\eta\alpha}+\frac{\overline{\delta-\tau\eta+\eta}\times\gamma}{\overline{\mu-\tau+2}\times\eta\alpha}\mathrm{A}+\frac{\overline{\delta-\tau\eta+2\eta}\times\gamma}{\overline{\mu-\tau+3}\times\eta\alpha}\mathrm{B}$$

$$+\ \frac{\overline{\delta-\tau\eta+3\eta}\times\gamma}{\overline{\mu-\tau+4}\times\eta\alpha}\mathrm{C}+\text{\&c. till }\frac{\overline{\delta-\eta}\times\gamma}{\mu\eta\alpha}\mathrm{F}.$$

$\phi =$ Fluent of $\overline{\alpha+\beta z^{\eta}}^{\,\mu} z^{\pi}\dot{z}$.

$\gamma = \alpha + \beta z^{\eta}$.

$\delta = \pi + 1 + \mu\eta$.

A, B, C *&c.* each preceding Term with its Sign.

Forms.	*Fluxions.*
15	$\overline{a+\beta z^{\eta}}^{\lambda} z^{\tau}\dot{z}$ *This Series will terminate when* $\frac{\tau+1}{\eta}+\lambda$ *is any negative whole Number.* N. B. *This Form fails when* $\frac{\tau+1}{\eta}$ *is any negative whole Number whatsoever.*
16	$\overline{a+\beta z^{\eta}}^{\frac{\lambda}{\nu}} z^{\tau}\dot{z}$ *This Series will terminate when* $\frac{\lambda}{\nu}$ *is any affirmative whole Number.*
17	$\overline{a+\beta z^{\eta}}^{\lambda} z^{\tau}\dot{z}$ × into $e+fz^{\eta}+gz^{2\eta}+hz^{3\eta}$ &c. *Here* λ *and* τ *must each be greater than* -1. *Also* z *and* $\overline{a+\beta z^{\eta}}^{\lambda+1}$ *must be one of them* $=0$ *at the beginning of the Fluent, the other* $=0$ *at the End of it.* *This Series will terminate when* $\frac{\tau+1}{\eta}$ *is any negative whole Number whatever.*

Fluents.

$$\frac{z^{\varepsilon}}{\varepsilon\alpha} - \frac{\delta+n}{\varepsilon+n}Aq - \frac{\delta+2n}{\varepsilon+2n}Bq - \frac{\delta+3}{\varepsilon+3n}Cq$$

$$- \frac{\delta+4n}{\varepsilon+4n}Dq - \&c. \times \text{ into } \overline{\alpha+\beta z^{n}}^{\mu+1}.$$

$\varepsilon = \pi + 1.$

$\delta = \varepsilon + \mu n.$

$q = \frac{\beta z^{n}}{\alpha}.$

A, B, C *&c.* each preceding Term with its Sign.

$$\frac{\alpha^{\frac{\mu}{\nu}} z^{\pi+1}}{\pi+1} + \frac{\frac{\mu}{\nu}Aq}{\pi+1+n} + \frac{\frac{\mu-\nu}{2\nu}Bq}{\pi+1+2n} + \frac{\frac{\mu-2\nu}{3\nu}Cq}{\pi+1+3n}$$

$$+ \frac{\frac{\mu-3\nu}{4\nu}Dq}{\pi+1+4n} + \&c.$$

$q = \frac{\beta z^{n}}{\alpha}.$

A, B, C *&c.* are the Numerators of each preceding Term with its Sign.

$$\phi \times : e - \frac{\varepsilon\alpha}{\delta\beta} \times f - \frac{\overline{\varepsilon+n} \times \alpha}{\overline{\delta+n} \times \beta}Ag - \frac{\overline{\varepsilon+2n} \times \alpha}{\overline{\delta+2n} \times \beta}Bh$$

$$- \frac{\overline{\varepsilon+3n} \times \alpha}{\overline{\delta+3n} \times \beta}Ck, \&c.$$

$\varepsilon = \pi + 1.$

$\delta = \pi + 1 + \overline{1+\mu} \times n.$

$\phi =$ whole Fluent of $\overline{\alpha+\beta z^{n}}^{\mu} z^{\pi} \dot{z}.$

A, B, C *&c.* each preceding Term with its Sign, but without the Quantities *f*, *g*, *h*, &c.

When any of the Quantities *e*, *f*, *g*, &c. are want- the reſpective Terms will vaniſh.

Forms.	*Fluxions.*
18	$e+fz^{n}+gz^{2n}+hz^{3n}\ \&c \times$ into $\overline{\alpha+\beta z^{n}+\gamma z^{2n}+\delta z^{3n}\ \&c}^{\,\mu}\ z^{\pi}\dot{z}.$ N. B. *This Form will fail when* $\frac{\pi+1}{n}$ *is any negative whole Number whatever, or when any Denominator is* 0.
19	$\overline{mz\dot{y}+ny\dot{z}}\times z^{n-1}y^{m-1}$ or $\overline{\frac{m\dot{y}}{y}+\frac{n\dot{z}}{z}}\times z^{n}y^{m}.$
20	$rzy\dot{v}+mzv\dot{y}+nyv\dot{z}\ \times$ into $z^{n-1}y^{m-1}v^{r-1}.$

Fluents.
$+\frac{e}{px}$ $+\frac{f\frac{-p}{-mn}\Big\}\beta A}{\overline{p+n}\times\alpha}z^{n}$ $+\frac{g\frac{-p}{-2mn}\Big\}\gamma A\frac{-\overline{p+n}}{-mn}\Big\}\beta B}{\overline{p+2n}\times\alpha}z^{2n}$ $+\frac{h\frac{-p}{-3mn}\Big\}\delta A\frac{-\overline{p+n}}{-2mn}\Big\}\gamma B\frac{-\overline{p+2n}}{-mn}\Big\}\beta C}{\overline{p+3n}\times\alpha}z^{3n}$ $+\frac{i\frac{-p}{-4mn}\Big\}\varepsilon A\frac{-\overline{p+n}}{-3mn}\Big\}\delta B\frac{-\overline{p+2n}}{-2mn}\Big\}\gamma C\frac{-\overline{p+3n}}{-mn}\Big\}\beta D}{\overline{p+4n}\times\alpha}z^{4n}$ $+$ &c. × into $z^{p}\times\overline{\alpha+\beta z^{n}+\gamma z^{2n}+\delta z^{3n}\ \&c.}^{\mu+1}$ $p=\pi+1.$ $m=\mu+1.$ A, B, C &c. the Coefficients of the 1ſt, 2d, 3d &c. Terms reſpectively.
$z^{n}y^{m}.$
$z^{n}y^{m}v^{r}.$

Forms.	*Given Fluxions.*
21	$\overline{e+fz^{\theta}}^{m}\, z^{\lambda\theta-1}\dot{z}$ *λ is any whole Number whatever.*
22	$\dfrac{z^{\lambda\theta-1}\dot{z}}{\overline{e+fz^{\theta}}\times\overline{g+hz^{\theta}}}$
23	$\overline{e+fz^{\theta}}^{m}\times\overline{g+hz^{\theta}}^{r}\, z^{\lambda\theta-1}\dot{z}$ *Here λ is any whole Number greater than* 0; *r is half of any whole Number whatever.*
24	$\dfrac{\overline{e+fz^{\theta}}^{m}\, z^{\lambda\theta-1}\dot{z}}{\overline{g+hz^{\theta}}^{m+r}}$ *Here either λ or $r-\lambda$ muſt be a poſitive whole Number greater than* 0.

Transform'd into others.
$\overline{v-e}^{\lambda-1} \times \frac{v^m \dot{v}}{\theta f^{\lambda}}$. $v = e + fz^{\theta}$.
$\frac{fz^{\lambda\theta-1}\dot{z}}{p \times \overline{e+fz^{\theta}}} - \frac{hz^{\lambda\theta-1}\dot{z}}{p \times \overline{g+hz^{\theta}}}$. $p = fg - eh$.
$\overline{v-e}^{\lambda-1} \times \overline{p + \frac{hv}{f}}\Big\rvert^{r} \times \frac{v^m \dot{v}}{\theta f^{\lambda}}$. $p = g - \frac{eh}{f}$. $v = e + fz^{\theta}$.
$\overline{gv-e}^{\lambda-1} \times \frac{\overline{f-hv}^{r-\lambda-1}\, v^m \dot{v}}{\theta p^{r-1}}$. $p = fg - eh$. $v = \frac{e+fz^{\theta}}{g+hz^{\theta}}$.

Forms.	Fluxions.
25	$\dfrac{\overline{e+fz^{\theta}}\vert^{m} z^{\theta-1}\dot{z}}{\overline{k+lz^{\theta}}\times\overline{g+hz^{\theta}}\vert^{m+r}}$ *Here r is any affirmative whole Number whatſoever.*
26	$\dfrac{z^{\lambda\theta-1}\dot{z}}{e+fz^{\theta}+gz^{2\theta}}$ N. B. *This Form fails when* 4*eg is greater than ff.*
27	$\overline{e+fz^{\theta}+gz^{2\theta}}\vert^{m} z^{\lambda\theta-1}\dot{z}$ λ *is any poſitive whole Number greater than* 0; *m is the half of any whole Number.*
28	$\dfrac{z^{\lambda\theta-1}\dot{z}}{e+fz^{\theta}+gz^{2\theta}}$ *Here e and g are affirmative.* λ *is the half of any affirmative whole Number greater than* 1. N. B. *This Form fails when ff is greater than* 4*eg.*

Transform'd.
$\frac{\overline{f-hv}^{r}}{\theta l b^{r}} \times \frac{av^{m}\dot{v}}{d+av} + \frac{\overline{f-hv}^{r-1}\, hv^{m}\dot{v}}{\theta l b^{r}}.$ $a = gl - kh.$ $b = fg - eh.$ $d = fk - el.$ $v = \frac{e+fz^{\theta}}{g+hz^{\theta}}.$
$\frac{g}{p} \times \frac{z^{\lambda\theta-1}\dot{z}}{\frac{f-p}{2}+gz^{\theta}} - \frac{g}{p} \times \frac{z^{\lambda\theta-1}\dot{z}}{\frac{f+p}{2}+gz^{\theta}}.$ $p = \sqrt{ff-4eg}.$
$\frac{\overline{v-\frac{1}{2}f}^{\lambda-1}}{\theta g^{\lambda}} \times \overline{p+\frac{v^{2}}{g}}^{m}\, \dot{v}.$ $p = e - \frac{ff}{4g}.$ $v = \frac{1}{2}f + gz^{\theta}.$
$\frac{\overline{v+\frac{1}{2}s}^{2\lambda-2}}{\theta s g^{2\lambda-3}} \times \frac{\dot{v}}{p+v^{2}} - \frac{\overline{V-\frac{1}{2}s}^{2\lambda-2}}{\theta s g^{2\lambda-3}} \times \frac{\dot{V}}{p+V^{2}}.$ $p = \frac{1}{4}fg + \frac{1}{2}g\sqrt{eg}.$ $s = \sqrt{2g\sqrt{eg}-fg}.$ $v = -\frac{1}{2}s + gz^{\frac{1}{2}\theta}.$ $V = +\frac{1}{2}s + gz^{\frac{1}{2}\theta}.$

Forms.	*Fluxions.*
29	$\dfrac{z^{\lambda\theta-1}\dot{z}}{\overline{h+kz^{\theta}}\times\overline{e+fz^{\theta}+gz^{2\theta}}}$ N. B. *This Form fails when* 4*eg is greater than ff.*
30	$\dfrac{z^{\lambda\theta-1}\dot{z}}{\overline{k+lz^{\theta}}\times\overline{e+fz^{\theta}+gz^{2\theta}}}$ *Here* λ *is any affirmative whole Number greater than* 0.
31	$\dfrac{z^{\lambda\theta-1}\dot{z}}{\overline{k+lz^{\theta}}\times\overline{e+fz^{\theta}+gz^{2\theta}}}$ *Here e and g are affirmative.* λ *is the half of any affirmative whole Number greater than* 2. N. B. *This Form fails when ff is greater than* 4*eg.*

Transform'd.

$$\frac{gkkz^{\lambda\theta-1}\dot{z}}{cd.\overline{h+kz^{\theta}}} + \frac{ggz^{\lambda\theta-1}\dot{z}}{dp.\overline{b+gz^{\theta}}} - \frac{ggz^{\lambda\theta-1}\dot{z}}{pc.\overline{a+gz^{\theta}}}.$$

$p = \sqrt{ff-4eg}$.

$a = \dfrac{f-p}{2}$.

$b = \dfrac{f+p}{2}$.

$c = ak - gh$.

$d = bk - gh$.

$$\frac{llz^{\lambda\theta-1}\dot{z}}{t.\overline{k+lz^{\theta}}} + \frac{\overline{v-\frac{1}{2}f}\,^{\lambda-1}}{\theta tg^{\lambda-1}} \times \frac{a\dot{v}-lv\dot{v}}{p+v^{2}}.$$

$a = gk - \frac{1}{2}lf$.

$p = eg - \frac{1}{4}ff$.

$t = ell - fkl + gkk$.

$v = \frac{1}{2}f + gz^{\theta}$.

$$-\frac{lkz^{\lambda\theta-\theta-1}\dot{z}}{t.\overline{k+lz^{\theta}}} + \frac{\overline{v+\frac{1}{2}s}\,^{2\lambda-3}}{2\theta stg^{2\lambda-3}} \times \frac{-2qv\dot{v}+rs\dot{v}}{p+v^{2}}$$

$$+\frac{\overline{V-\frac{1}{2}s}\,^{2\lambda-3}}{2\theta stg^{2\lambda-3}} \times \frac{2qV\dot{V}+rs\dot{V}}{p+V^{2}}.$$

$p = \frac{1}{2}g\sqrt{eg} + \frac{1}{4}fg$.

$q = l\sqrt{eg} - gk$.

$r = l\sqrt{eg} + gk$.

$s = \sqrt{2g\sqrt{eg} - fg}$.

$t = ell - fkl + gkk$.

$v = -\frac{1}{2}s + gz^{\frac{1}{2}\theta}$.

$V = +\frac{1}{2}s + gz^{\frac{1}{2}\theta}$.

Forms.	*Fluxions.*
32	$\dfrac{z^{\theta-1}\dot{z}}{\overline{k+lz^{\theta}} \times \overline{e+fz^{\theta}+gz^{2\theta}}^{m}}$ *Here m is half of any poſitive whole Number.*
33	$\dfrac{z^{\lambda\theta-1}\dot{z} \times \overline{e+fz^{\theta}}^{m}}{h+kz^{\theta}+lz^{2\theta}}$ *Here* λ *is any affirmative whole Number greater than* 0. N. B. *This Form fails when* 4*hl is greater than kk.*

Transform'd.

$$\frac{\overline{lv - \frac{1}{2}b}^{\,2m-1}}{-n} \times \frac{\dot{v}}{\overline{ap + av^2}^{\,m}}.$$

$$a = ell - fkl + gkk.$$

$$b = fl - 2gk.$$

$$p = eg - \tfrac{1}{4}ff.$$

$$v = \frac{el - \frac{1}{2}fk + \frac{1}{2}bz^{\theta}}{k + lz^{\theta}}.$$

$$\frac{l \times \overline{v - e}^{\,\lambda-1}}{\theta p f^{\lambda-1}} \times \frac{v^m \dot{v}}{\frac{k-p}{2}f - le + lv}$$

$$- \frac{l \times \overline{v - e}^{\,\lambda-1}}{\theta p f^{\lambda-1}} \times \frac{v^m \dot{v}}{\frac{k+p}{2}f - le + lv}.$$

$$p = \sqrt{kk - 4hl}.$$

$$v = e + fz^{\theta}.$$

Forms.	*Fluxions.*
34	$\dfrac{z^{\lambda\theta-1}\dot{z}\times\overline{e+fz^{\theta}}^{m}}{h+kz^{\theta}+lz^{2\theta}}$ *Here h and l are affirmative.* λ *is any poſitive whole Number greater than* 0. *m is the half of any affirmative odd Number.* N. B. *This Form fails when kk is greater than 4hl, or when a is negative.*
35	$\dfrac{z^{\lambda\theta-1}\dot{z}}{\overline{h+kz^{\theta}+lz^{2\theta}}\times\overline{e+fz^{\theta}}^{m}}$ *Here h and l are affirmative.* λ *is any poſitive whole Number leſs than* $m+1$. *m is the half of any poſitive odd Number.* N. B. *This Form will fail when kk is greater than 4hl, or when a is negative.*

Transform'd.

$$\frac{\overline{vv - sv + q}^{\lambda-1} \times \overline{v + \tfrac{1}{2}s}^{2m}}{\theta s f^{\lambda-2}\, l^{2\lambda+2m-3}} \times \frac{\dot{v}}{p + vv}$$

$$- \frac{\overline{VV + sV + q}^{\lambda-1} \times \overline{V - \tfrac{1}{2}s}^{2m}}{\theta s f^{\lambda-2}\, l^{2\lambda+2m-3}} \times \frac{\dot{V}}{p + VV}.$$

$a = ffh - efk + eel.$
$b = fk - 2el.$
$s = \sqrt{2l\sqrt{al} - bl}.$
$p = l\sqrt{al} - \frac{1}{4}ss.$
$q = \frac{1}{4}ss - ell.$
$v = -\frac{1}{2}s + l\sqrt{e + fz^{\theta}}.$
$V = +\frac{1}{2}s + l\sqrt{e + fz^{\theta}}.$

$$+ \frac{\overline{-eVV + esV + q}^{\lambda-1} \times \overline{V - \tfrac{1}{2}s}^{2m-2\lambda+2}}{\theta s f^{\lambda-2} a^{m} l^{m}} \times \frac{l\dot{V}}{p + V^2}$$

$$- \frac{\overline{-evv - esv + q}^{\lambda-1} \times \overline{v + \tfrac{1}{2}s}^{2m-2\lambda+2}}{\theta s f^{\lambda-2} a^{m} l^{m}} \times \frac{l\dot{v}}{p + v^2}.$$

$a = ffh - efk + eel.$
$b = fk - 2el.$
$s = \sqrt{2l\sqrt{al} - bl}.$
$p = l\sqrt{al} - \frac{1}{4}ss.$
$q = al - \frac{1}{4}sse.$
$v = -\frac{1}{2}s + \sqrt{\frac{al}{e + fz^{\theta}}}.$
$V = +\frac{1}{2}s + \sqrt{\frac{al}{e + fz^{\theta}}}.$

The Uſe of the foregoing Table of Fluxions.

1. WHEN it is required to find the Fluent of a given Fluxion, by this Table; it muſt firſt be found out to what *Form* it belongs. To this End *ſubſtitute* η or θ (as the Caſe requires) for the *numeral Index*, and then it may be compared with the ſeveral *Forms* in the Table. Or the given Fluxion may (and often muſt) be *reduced* to another Expreſſion, by dividing it by the higheſt Power in the Radical, (that is by taking the higheſt Power of the variable Quantity out of the Radical) and affecting the other Quantities with it; by which Means the Fluxion will acquire a new Form and Expreſſion, for the Sign of the Index will be changed, *(and this I call reducing the Index)*. Then ſubſtitute η or θ for the numeral Index, and ſee what Form in the Table it will agree to.

For Example, let $\frac{\dot{x}}{\overline{aa+xx}\sqrt{bb-xx}}$ be given; here x^2 will be x^θ, $\theta=2$, and $\frac{1}{2}\theta=1$; and $\dot{x}$ or $x^{1-1}\dot{x}$ $=x^{\frac{1}{2}\theta-1}\dot{x}$; therefore the Fluxion is $\frac{x^{\frac{1}{2}\theta-1}\dot{x}}{\overline{aa+x^\theta}\sqrt{bb-x^\theta}}$; then comparing this with the *Twenty-third Form*, we ſhall have $\lambda=\frac{1}{2}$, which ought to be a whole Number, and therefore it comes not under this Form. Now reduce the Index by dividing by xx, and we ſhall have $\frac{\dot{x}}{x^3\times\overline{1+aax^{-2}}\sqrt{-1+bbx^{-2}}}=$ $\frac{x^{-2-1}\dot{x}}{\overline{1+a^2x^{-2}}\sqrt{-1+bbx^{-2}}}$; where θ will be $=-2$, and the Index $-2-1=\theta-1$, therefore $\lambda=1$; and therefore it belongs to Form 23.

2. If the Fluxion, then, is a Binomial, you muſt firſt try if the Fluent of either Expreſſion can be had in *finite Terms* by Form 3d or 15th, (which may always be known by the Notes in the ſecond Column); if it cannot, then try whether it can be had in finite Terms by any other of the Forms for Binomials, which will be eaſy by comparing the Indices. If it fall under the 11th, 12th, 13th, or 14th Form, there will be required two (or perhaps three) Operations, whereof the *firſt* is always for the Fluent of the *original Fluxion*, and is to be found by ſome of the firſt ten Forms. But if it cannot be found in finite Terms by theſe Forms, then the *laſt Recourſe* is to the 15th, 16th, 17th, or 18th Form, as beſt ſuits the Caſe.

3. Having found to what Form (or Forms) the Fluxion belongs, you have no more to do, but only to *write* the reſpective *Values* of the *general Quantities* in the Fluent belonging to that *Form*, and multiply the *Reſult* by ſuch given Quantities as the given *Fluxion* was multiply'd into, and you have the *Fluent*.

4. And in compound Binomials, or Trinomials, ſuch as belong to the 21ſt and following Forms, ſince theſe are transform'd into ſingle binomial Fluxions, ſtanding in the third Column; therefore you muſt proceed with theſe according to the foregoing Directions for Binomials; and the Fluents of theſe (ſingle) Binomials being found, will be the Fluents of the Trinomial or compound Binomial Fluxions ſtanding in the ſecond Column, reſpectively.

5. But there are ſeveral Sorts of compound Fluxions which cannot be reſolved without ſome further Reduction. Now there are theſe two other Ways of reducing a given Fluxion to a different Form or Expreſſion. The firſt is actually to multity by any Power of the Quantity under the Vinculum (in any Binomial or Trinomial Surd) and then leſſen the Index of that Surd by the ſame Power if it is in the

Numerator,

Numerator, or increaſe it in the Denominator: This alters the Power of the ſurd Quantity. The other Way is actually to divide by the Quantity under the Vinculum, and then leſſen the Index of that Surd by 1 if it be in the Denominator, or increaſe it by 1 if it is in the Numerator: This alters the Dimenſion of the ſimple variable Quantity. And in both you will have the more Terms according as you multiply by a greater Power, or continue your Diviſion the further. But the laſt Term only will be of a like Form with the given Fluxion, differing only in the Index of the Surd, or of the variable Quantity.

6. Therefore in any Fluxion, but eſpecially in compound Binomials and Trinomials, if there be a Surd in the Numerator you may *leſſen* its Dimenſion at Pleaſure (or take it quite out of the Numerator) if you *multiply* by ſome *Power* of the Quantity under the Vinculum, and *leſſen* its Index by the *ſame* Power.

7. Alſo in any given Fluxion, if the Index of the ſimple variable Quantity be too high, (as 2θ, 3θ, *r*θ, *&c.*), or when it is too low or negative, (as — , — θ, — 2θ, *&c.*) it may be alter'd at Pleaſure, by *dividing* by ſome rational Compound Quantity in the Denominator if there is any, or elſe by the Quantity under the Vinculum, and *ſubtracting* 1 from its Index, if it is in the *Denominator*, or adding 1 in the *Numerator*; and continuing the Diviſion to a proper Number of Places.

For Example, if $\frac{\dot{x}\sqrt{bb-xx}}{aa+xx}$ be given; reduce the Index, and it becomes $\frac{x^{-1}\dot{x}\sqrt{-1+bbx^{-2}}}{1+aax^{-2}}$, but neither Way will it agree with any of the Forms; for in the firſt Caſe $\theta=2$, and $\dot{x}=x^{1-1}\dot{x}=x^{\frac{1}{2}\theta-1}\dot{x}$, which makes $=\frac{1}{2}$; and in Caſe 2d, $=-2$, and $x^{-1}\dot{x}=x^{0-1}\dot{x}$, and $\lambda=0$; but λ ought to be a whole

Number. Therefore (by Art. 6.) I multiply both Numerator and Denominator of $\frac{\dot{x}\sqrt{bb-xx}}{aa+xx}$ by $\sqrt{bb-xx}$, and it becomes $\frac{bb\dot{x}-x^2\dot{x}}{\overline{aa+xx}\,\sqrt{bb-xx}}$; now the firſt Term $\frac{bb\dot{x}}{\overline{aa+xx}\,\sqrt{bb-xx}}$ comes under the 23d Form, being of the ſame Form as that in Art. 1. And the latter Term $\frac{-x^2\dot{x}}{\overline{aa+xx}\,\sqrt{bb-xx}}$ being actually divided by $xx+aa$ (*according to Art.* 7.) it becomes $\frac{-\dot{x}}{\sqrt{bb-xx}}+\frac{aa\dot{x}}{\overline{aa+xx}\,\sqrt{bb-xx}}$, the latter Term (like that Art. 1.) belongs to Form 23d; and the former to Form 10.

Or thus.

Divide $\frac{\dot{x}\sqrt{bb-xx}}{aa+xx}$ by $aa+xx$, and it becomes $\frac{\dot{x}\sqrt{bb-xx}}{xx}-\frac{aa\dot{x}\sqrt{bb-xx}}{xx\times\overline{aa+xx}}$. Reduce the Index in the latter Term, and it is $\frac{-aax^{-2-1}\dot{x}\sqrt{bb-xx}}{1+aax^{-2}}$, which belongs to the 23d or 24th Form, where $\lambda=1$; and the firſt Part belongs to Form 12.

By theſe Operations the given Fluxion is reduced to ſeveral Terms; all which (except the laſt) muſt be ſtill further reduced if there be Occaſion, by repeating the ſame Method, till at laſt all the reſulting Terms will fall under ſome or other of the Forms in the Table; as in the Examples here given, and many more will follow afterwards. By theſe Sorts of Reduction, a given Fluxion is prepared for a Solution, when it does not fall directly at firſt, under any of the Forms.

8. In *Form* 16, when the Denominator of any Term happens to 0, that particular Term muſt be found thus; take the *known Part* of the Numerator (that is leaving out the Powers of z), and *multiply* it into 2.302585 *Log.* z, and you have that Term. And the ſaid known Part of the Numerator will be the Value of the Capital Letter in the next following Term.

For Example, In finding the Fl: $x^3\dot{x}\sqrt{1+\frac{1}{xx}}$, we have this Series $\frac{x^4}{4} + \frac{\frac{1}{2}A\frac{1}{xx}}{2} - \frac{\frac{1}{4}B\frac{1}{xx}}{0} + \frac{\frac{1}{2}C\frac{1}{xx}}{2} - \frac{\frac{5}{8}D\frac{1}{xx}}{4}$ *&c.* Inſtead of the third Term, take $-\frac{1}{4}B \times \text{Log}:x$. But $B = \frac{1}{2}A = \frac{1}{2}$ (without x); therefore $-\frac{1}{4}B = -\frac{1}{8} = C$. Then the Series is $\frac{x^4}{4} + \frac{\frac{1}{2}A\frac{1}{xx}}{2} - \frac{1}{8} \times 2.30258\,\text{Log}:x + \frac{\frac{-1\times 1}{8\times 2xx}}{2} - \frac{\frac{5}{8}D\frac{1}{xx}}{4}$ *&c.*

9. Though theſe Forms contain Variety of Logarithmic Fluents; yet each of them may be changed ſeveral Ways into different Quantities. Thus, when a Logarithm is in the Fluent, you may multiply (the *Number* whoſe Log. is there) and then divide it by ſome compound Quantity, which you ſee will make it ſimpler: Or you may ſquare it and take half the Logarithm: Or you may extract the ſquare Root and take double the Logarithm: Or you may multiply it or divide it by any *given* Quantity: Or make the Numerator and Denominator change Places, and

then change the Sign of the Logarithm, &c. And thus you may always find the ſimpleſt Expreſſion for the Logarithmic Quantity.

Thus, $\text{Log:} \frac{x}{y} = \text{Log:} \frac{ax}{ay} = \frac{1}{2}\text{Log.} \frac{xx}{yy} = 2\,\text{Log:} \sqrt{\frac{x}{y}} = -\text{Log:} \frac{y}{x}$, &c.

10. And if you have an untractable Fluxion that will anſwer to none of the Forms; it may ſometimes be *transformed* into others, by Prop. 9, which may then be reſolved by the Table.

The following Examples will make the Proceſs very plain.

EXAMPLE I.

To find the Fluent of $\frac{x\dot{x}}{xx - aa}$.

Here $\eta = 2$, by which expunging the numeral Index, the Fluxion will be reduced to $\frac{x^{\eta-1}\dot{x}}{-aa + x^{\eta}}$, which is a Fluxion belonging to the 4th Form. Therefore $\alpha = -aa$, $\beta = 1$, $z = x$, and $\frac{2.3025}{\eta\beta} \times \text{Log. of } \overline{\alpha + \beta z^{\eta}} = \frac{2.3025}{2} \times \text{Log. } \overline{xx - aa} = 2.3025851\ \text{Log. } \sqrt{x^2 - a^2}$, the Fluent required.

Ex. 2.

Let the Fluxion $\frac{r\dot{x}}{\sqrt{bb + xx}}$ *be given.*

Here $\eta = 2$; whence the Fluxion will be reduced to $\frac{rx^{\frac{1}{2}\eta - 1}\dot{x}}{\sqrt{bb + x^{\eta}}}$, a Fluxion of the 9th Form; whence $\alpha = bb$, $\beta = 1$, $z = x$, and $\varphi = 2.30258\,\text{Log.} \overline{x + \sqrt{bb + xx}}$ =

$= \text{Fluent of } \frac{\dot{x}}{\sqrt{bb + xx}}$; therefore the Fluent o

$$\frac{r\dot{x}}{\sqrt{bb + xx}} = 2.302585r \times \text{L}: \overline{x + \sqrt{bb + xx}}.$$

Ex. 3.

To find the Fluent of $\frac{bx^{\frac{1}{3}}\dot{x}}{\overline{c - ax^{\frac{2}{3}}}|^{\frac{3}{5}}}$.

Here $n = \frac{2}{3}$, and the Fluent becomes $\overline{c - ax^{n}}|^{-\frac{3}{5}} \times bx^{2n-1}\dot{x}$, which belongs to Form the 3d and 11th. Or rather thus, by taking $x^{\frac{2}{3}}$ out of the Radical, the Fluxion becomes $\overline{-a + cx^{-\frac{2}{3}}}|^{-\frac{3}{5}} \times bx^{-\frac{1}{15}}\dot{x}$, which belongs to form 15th. Here $\alpha = -a$, $\beta = c$, $z = x$, $n = -\frac{2}{3}$, $\mu = -\frac{3}{5}$, $\pi = -\frac{1}{15}$, $\varepsilon = \frac{14}{15}$, $\delta = \frac{4}{3}$. Then ſince $\frac{\pi + 1}{n} + \mu = -2$ a negative whole Number, therefore the Fluent will be had in finite Terms, and is $= b \times : -\frac{15x^{\frac{14}{15}}}{14a} - \frac{75cx^{\frac{4}{15}}}{28aa} : \times \overline{-a + cx^{-\frac{2}{3}}}|^{\frac{2}{5}}$

$= -\frac{30abx^{\frac{2}{3}} + 75bc}{28qa} \times \overline{c - ax^{\frac{2}{3}}}|^{\frac{2}{5}}$, the Fluent required.

Ex. 4.

To find the Fluent of $\frac{b\dot{z}}{z\sqrt{zz - cc}}$.

This belongs to Form the 12th, having by Form the 3d the Fluent of $\frac{bz\dot{z}}{\sqrt{zz - cc}}$; in which Caſe, $\pi = 1$, $n = 2$, and $-1 = 1 - 2 = \pi - n$, and the given Fluxion becomes $\frac{bz^{\pi-n}\dot{z}}{\sqrt{-c^2+z^n}}$. But ſince $\frac{+1}{n}=1$, therefore the Fluent cannot be had by Form the 12th. Wherefore I reduce it to $\frac{bz^{-2}\dot{z}}{\sqrt{1-ccz^{-2}}}$, which is a Fluxion of the 10th Form, where $n=-2$, $\alpha=1$, $\beta=-cc$; and the Fluent $=-\frac{b}{c}\times,017453$ Degrees of the Arch whoſe Sine s $\frac{c}{z}$.

Ex. 5.

Let the Fluent of $\frac{a\dot{x}}{x}\sqrt{2ax+xx}$, or $ax^{\frac{1}{2}-1}\dot{x}\sqrt{2a+x}$ *be required.*

This belongs to the 9th and 13th Forms. To get the Fluent of $\frac{x^{\frac{1}{2}-1}\dot{x}}{\sqrt{2a+x}}$ by Form 9th; here $=2a$, $\beta=1$, $z=x$, $n=1$: Whence the Fluent $=2.3025\text{Log}:\overline{a+x+\sqrt{2ax+xx}}:=\varphi$. The given Fluxion therefore would become $ax^{\pi}\dot{x}\times\overline{2a+x}|^{\mu+\tau}$. Where $\pi=-\frac{1}{2}$, $\mu=-\frac{1}{2}$, $\tau=1$, $\alpha=2a$, $\beta=1$, $=1$, $z=x$, $\gamma=2a+x$, $\delta=0$: And the Fluent of $x^{-\frac{1}{2}}\dot{x}\sqrt{2a+x}$ is $=a\varphi+x^{\frac{1}{2}}\sqrt{2a+x}$, by Form 13. And the Fluent of $ax^{-\frac{1}{2}}\dot{x}\sqrt{2a+x}=aa\varphi+\sqrt{2ax+xx}$.

Ex.

Ex. 6.

To find the Fluent of $\frac{bz^2\dot{z}}{aa-\frac{aa}{bb}z^2}$.

This will be had by Form the 6th and 11th

To find the Fluent of $\frac{\dot{z}}{aa-\frac{aa}{bb}z^2}$ by Form 6th

Here $\alpha = aa$, $\beta = \frac{aa}{bb}$, $n = 2$, then the Fluent is =

$\frac{b}{2aa} \times 2.3025$ Log. $\frac{a+\frac{a}{b}z}{a-\frac{a}{b}z} = \frac{b}{aa} \times 2.3025 \times$ Log

$\sqrt{\frac{b+z}{b-z}} = \varphi$. Then by Form 11th, $\alpha = aa$, $\beta =$

$-\frac{aa}{bb}$, $\mu = -1$, $\pi = 0$, $\lambda = 1$, $\gamma = 1$, $\varepsilon = 3$, $\delta =$

whence the Fluent of $\frac{z^2\dot{z}}{aa-\frac{aa}{bb}z^2}$ is $= \frac{-aa \times 1}{-\frac{aa}{bb} \times 1}$

$+ \frac{bbz}{-aa} = bb\varphi - \frac{bb}{aa}z$; and conſequently the Fluent of $\frac{bz^2\dot{z}}{aa-\frac{aa}{bb}z^2}$ is $= b^3\varphi - \frac{b^3}{aa}z = \frac{b^4}{aa}$

2.3025 Log. $\sqrt{\frac{b+z}{b-z}} : -\frac{b^3}{aa}z$.

Ex. 7.

To find the Fluent of $b\dot{x}\sqrt{\dfrac{4a-3x}{a-x}}$.

This by Form 23d is transformed into $b\times\overline{a+3v}^{\frac{1}{2}}\times\dfrac{v^{-\frac{1}{2}}\dot{v}}{-1}$. Where $\lambda=1$, $e+fz^{\theta}=a-x=v$, $g+bz^{\theta}=4a-3x$, $p=a$: By Form the 9th the Fluent of $\dfrac{v^{-\frac{1}{2}}\dot{v}}{\sqrt{a+3v}}$ is $\dfrac{2}{\sqrt{3}}\times 2.30258\times\text{Log.}\ \overline{\sqrt{3v}+\sqrt{a+3v}}:=\varphi$. And by Form 13th, Fluent of $-bv^{-\frac{1}{2}}\dot{v}\sqrt{a+3v}$ is $=-\dfrac{ba}{2}\varphi-bv^{\frac{1}{2}}\sqrt{a+3v}=-b\sqrt{\overline{a-x}\times\overline{4a-3x}}-\dfrac{2.30258ba}{\sqrt{3}}\times\text{Log:}\ \overline{\sqrt{3a-3x}+\sqrt{4a-3x}}$.

Ex. 8.

To find the Fluent of $\dfrac{x^{-1}\dot{x}\sqrt{aa-xx}}{bb+\frac{b}{a}xx}$.

This by Division is reduced to $\dfrac{x^{-1}\dot{x}}{bb}\sqrt{aa-xx}-\dfrac{x\dot{x}}{ba}\times\dfrac{\sqrt{aa-xx}}{bb+\frac{b}{a}xx}$. The Fluent of $\dfrac{x^{-1}\dot{x}}{bb}\sqrt{aa-xx}$ (by Form 9th and 13th) is $=\dfrac{\sqrt{aa-xx}}{bb}-\dfrac{2.30258a}{bb}\times\text{Log.}\ \dfrac{a+\sqrt{aa-xx}}{x}$. And by Form the 23d, $\dfrac{x\dot{x}\sqrt{aa-xx}}{bb+\frac{b}{a}xx}$ is transformed to $\overline{p+\frac{b}{f}v}\big|^{-1}\times\dfrac{v^{1}\dot{v}}{\theta f}$

$$=\frac{-\frac{1}{2}v^{\frac{1}{2}}\dot{v}}{bb+ba-\frac{b}{a}v}$$: For $g+hz^{\theta}=bb+\frac{b}{a}xx$, $e+fz^{\theta}$ $=aa-xx=v$, $p=bb+ba$, $\theta f=-2$, $m=\frac{1}{2}$, $r=-1$, $\lambda=1$. Now by Form the 6th, the Fluent of $\frac{v^{-\frac{1}{2}}\dot{v}}{p-\frac{b}{a}v}$

$$=2.30258\sqrt{\frac{a}{pb}}\times\text{Log. of }\frac{p+\frac{b}{a}v+2\sqrt{\frac{pbv}{a}}}{p-\frac{b}{a}v}=\varphi.$$

And by Form 11th the Fluent of $\frac{1}{2ba}\times\frac{v^{\frac{1}{2}}\dot{v}}{p-\frac{b}{a}v}$ $=\frac{p}{2bb}\varphi-\frac{\sqrt{v}}{bb}$: Whence the Fluent of $\frac{x^{-1}\dot{x}\sqrt{aa-xx}}{bb+\frac{b}{a}xx}$ is $=\frac{\sqrt{aa-xx}}{bb}-\frac{\sqrt{v}}{bb}+\frac{2.3025a}{bb}\text{Log.}\frac{a-\sqrt{aa-xx}}{x}$ $+\frac{p}{2bb}\varphi=\frac{2a\times 2.30258\text{ Log: }\frac{a-\sqrt{v}}{x}:+p\varphi}{2bb}.$

Ex. 9.

The Fluent of $bbx^{-2}\dot{x}\times\frac{\overline{a+x}|^{\frac{3}{2}}}{\sqrt{ab+ax}}$ *is required.*

Here $\theta=1$, and the Fluxion would become $bbx^{-\theta-1}\dot{x}\times\frac{\overline{a+x^{\theta}}|^{\frac{3}{2}}}{\sqrt{ab+ax^{\theta}}}$, therefore the given Fluxion is reduced by Multiplication to $\frac{bbaa\dot{x}+2bbax\dot{x}+bbx^2\dot{x}}{xx\sqrt{ab+ax}\times\sqrt{a+x}}$

=

$= \frac{bb\dot{x}}{\sqrt{ab+ax}\sqrt{a+x}} + \frac{2bbax^{-2}\dot{x}}{\sqrt{a+abx^{-1}}\sqrt{1+ax^{-1}}} + \frac{bba^2x^{-3}\dot{x}}{\sqrt{a+abx^{-1}}\sqrt{1+ax^{-1}}}$; each Term whereof belongs to the 23d Form. Let $e+fz^\theta = a+x = v$, $g+bz^\theta = ab+ax$, $\theta = 1$, $r = -\frac{1}{2}$, $m = -\frac{1}{2}$, $p = ab-aa$, then $\frac{bb\dot{x}}{\sqrt{ab+ax}\sqrt{a+x}} = \frac{bbv^{-\frac{1}{2}}\dot{v}}{\sqrt{p+av}}$, whoſe Fluent by Form the 9th will be $\frac{2bb}{\sqrt{a}} \times 2.30258$ Log: $\overline{\sqrt{av}+\sqrt{p+av}}: = \frac{2bb}{\sqrt{a}} \times 2.30258$ Log: $\overline{\sqrt{aa+ax}+\sqrt{ab+ax}}: = 2.302585 \times \frac{2bb}{\sqrt{a}} \times$ Log: of $\overline{\sqrt{a+x}+\sqrt{b+x}}: = \frac{2bb}{\sqrt{a}}$A, the Fluent of the firſt Term.

Again for the 2d and 3d Terms: Make $e+fz^\theta = 1+ax^{-1} = v$, $g+bz^\theta = a+abx^{-1}$, $\theta = -1$, $p = a-b$, $\lambda = 1$ in the ſecond Term, and $\lambda = 2$ in the third Term, which therefore will be transform'd into $-\frac{2bbv^{-\frac{1}{2}}\dot{v}}{\sqrt{p+bv}}$ and $\frac{bbv^{-\frac{1}{2}}\dot{v} - bbv^{\frac{1}{2}}\dot{v}}{\sqrt{p+bv}}$, whoſe Sum is $-\frac{bbv^{-\frac{1}{2}}\dot{v}}{\sqrt{p+bv}} - \frac{bbv^{\frac{1}{2}}\dot{v}}{\sqrt{p+bv}}$. By Form the 9th the Fluent of $\frac{v^{-\frac{1}{2}}\dot{v}}{\sqrt{p+bv}}$ $= \frac{2}{\sqrt{b}} \times 2.3025$ Log: $\overline{\sqrt{bv}+\sqrt{p+bv}}: = 2.30258 \times \frac{2}{\sqrt{b}} \times$Log: $\overline{\sqrt{\frac{ba+bx}{x}} + \sqrt{\frac{ba+ax}{x}}}: = \frac{2}{\sqrt{b}}$B. And by Form the 11th, the Fluent of $\frac{v^{\frac{1}{2}}\dot{v}}{\sqrt{p+bv}}$ is $= \frac{-p}{b\sqrt{b}}\text{B} + \frac{v^{\frac{1}{2}}}{b}\sqrt{p+bv}$, multiply the Sum of theſe Fluents by $-bb$, and you will have $\frac{\overline{-2bb+bp}}{\sqrt{b}}\times\text{B} - bv^{\frac{1}{2}}\sqrt{p+bv} = \frac{ba-3bb}{\sqrt{b}}\text{B} - \frac{b}{x}\sqrt{ab+ax}\,\sqrt{a+x}$, to which add the firſt Term $\frac{2bb}{\sqrt{a}}\text{A}$, and you will have $\frac{2bb}{\sqrt{a}}\text{A} + \frac{ba-3bb}{\sqrt{b}}\text{B} - \frac{b}{x}\sqrt{ab+ax}\sqrt{a+x}$, the Fluent required.

Ex. 10.

To find the Fluent of $\frac{\dot{z}}{4bz}\sqrt{a^2b^2 - a^3z + 16b^2z^2}$.

The given Fluxion will be reduced to $\frac{\dot{z}}{4bz} \times \frac{a^2b^2 - a^3z + 16b^2z^2}{\sqrt{a^2b^2 - a^3z + 16b^2z^2}} = \frac{a^2bz^{-2}\dot{z}}{4\sqrt{16b^2 - a^3z^{-1} + a^2b^2z^{-2}}} - \frac{a^3\dot{z}}{4b\sqrt{a^2b^2 - a^3z + 16b^2z^2}} + \frac{4bz\dot{z}}{\sqrt{a^2b^2 - a^3z + 16b^2z^2}}$, each of theſe Terms belongs to the 27th Form.

For the firſt Term, $v = -\frac{1}{2}a^3 + a^2b^2z^{-1}$, $p = 16bb - \frac{a^4}{4bb}$, $\lambda = 1$, $m = -\frac{1}{2}$, $\theta = -1$, and it will be transformed into $\frac{-\dot{v}}{4b\sqrt{p + \frac{vv}{g}}} = \frac{-a\dot{v}}{4\sqrt{pg + vv}}$, whoſe Fluent by Form the 9th will be $-\frac{a}{4} \times$ 2.30258 Logar: $v + \sqrt{pg + vv}$: $= \frac{-a}{4} \times 2.30258 \times$ Log: $\frac{a^2b^2}{z} - \frac{1}{2}a^3 + \frac{ab}{z}\sqrt{a^2b^2 - a^3z + 16b^2z^2}$.

Again for the other two Terms, here $v=-\frac{1}{2}a^3+16b^2z$, $p=a^2b^2-\frac{a^6}{64b^2}$, $m=-\frac{1}{2}$, $\theta=1$, $\lambda=1$ in the ſecond Term, and $=2$ in the third: And theſe two Terms by Form the 27th will be transformed into

$$\frac{-a^3\dot{v}}{16bb\sqrt{pg+vv}} \text{ and } \frac{v\dot{v}}{16bb\sqrt{pg+vv}}+\frac{a^3\dot{v}}{32bb\sqrt{pg+vv}},$$

whoſe Sum is $\frac{v\dot{v}}{16bb\sqrt{pg+vv}}-\frac{a^3\dot{v}}{32bb\sqrt{pg+vv}}$;

whoſe Fluent (by Form the 9th) $=\frac{\sqrt{pg+vv}}{16bb}-\frac{a^3L}{32bb}\times\text{Log. of}: v+\sqrt{pg+vv}: =\frac{\sqrt{a^2b^2-a^3z+16bbz^2}}{4b}$ $-2.302585\times\frac{a^3}{32bb}\times\text{Log}: -\frac{1}{2}a^3+16bbz+4b\sqrt{a^2b^2-a^3z+16b^2z^2}$: Whence the Fluent of $\frac{\dot{z}}{4bz}\sqrt{a^2b^2-a^3z+16b^2z^2}$ is $=\frac{1}{4b}\sqrt{a^2b^2-a^3z+16bbzz}$ $-\frac{a}{4}\times 2.302585\,\text{Log}: \frac{a^2b^2}{z}-\frac{1}{2}a^3+\frac{ab}{z}\times\sqrt{a^2b^2-a^3z+16b^2z^2}: -\frac{a^3}{32bb}\times 2.302585\,\text{Log}: 16bbz$ $-\frac{1}{2}a^3+4b\sqrt{a^2b^2-a^3z+16b^2z^2}$.

I ſhall add a few more Examples, chiefly to illuſtrate the Method of Reduction.

Ex. II.

Let $\frac{dz^{\frac{3}{5}-1}\dot{z}}{e+fz}$ *be propoſed:* Divide by z and it will be reduced to $\frac{dz^{-\frac{2}{5}-1}\dot{z}}{f+ez^{-1}}$. This belongs to Form 7

7th, where $n = -1$, $\lambda = 5$, $\delta = 2$, $x = \sqrt[5]{\frac{e}{fz}}$, $K = 36°$ $a = {,}587785$; $b = {,}951056$; $c = 0$, and $s = {,}809017$; $t = -{,}309017$; $u = -1$; where always obſerve that the Coſines of Arches above 90° are negative Numbers: then will be found P, Q, R in Degrees; and the Logarithms of $\sqrt{1 - 2sx + xx}$, $\sqrt{1 - 2tx + xx}$, $\sqrt{1 - 2ux + xx}$, for which Logarithms put S, T, V; Then (by Caſe 6th) the Fluent is $\frac{d}{-f^{\frac{3}{5}} e^{\frac{2}{5}}}$ × into $-{,}309017 \times 2LS + {,}951056 \times 2NP + {,}809017 \times 2LT - {.}587785 \times 2NQ - LV$.

Ex. 12.

Given $x^{-3}\dot{x} \times \overline{aa - x^2}^{\frac{2}{3}}$, this by Form 21ſt (putting $v = aa - x^2$) is transformed into $\overline{v - aa}^{-2} \times \frac{v^{\frac{2}{3}}\dot{v}}{-2} = -\frac{1}{2} \times \frac{v^{\frac{2}{3}}\dot{v}}{\overline{-aa + v}^{2}}$; which belongs to Form 7th and 14th.

Ex. 13.

Let $\frac{x^{-1}\dot{x} \times \overline{aa + xx}^{\frac{3}{2}}}{\overline{abb - bxx}^{\frac{3}{2}}}$ *be given*; this is reduced to $\frac{aax^{-1}\dot{x} \times \overline{aa + xx}^{\frac{1}{2}}}{\overline{abb - bxx}^{\frac{3}{2}}} + \frac{x\dot{x} \times \overline{aa + xx}^{\frac{1}{2}}}{\overline{abb - bx^2}^{\frac{3}{2}}}$, both which Terms belong to Form 24.

Ex. 14.

Let $\frac{x^2\dot{x}\times\overline{a+x}^{\frac{1}{3}}}{\overline{b+x}\times\overline{ab-bx}^{\frac{4}{3}}}$ *be propoſed*; Divide by $x+b$, and it is reduced to $\frac{x\dot{x}\times\overline{a+x}^{\frac{1}{3}}}{\overline{ab-bx}^{\frac{4}{3}}}-\frac{b\dot{x}\times\overline{a+x}^{\frac{1}{3}}}{\overline{ab-bx}^{\frac{4}{3}}}$ $+\frac{bb\dot{x}\times\overline{a+x}^{\frac{1}{3}}}{\overline{b+x}\times\overline{ab-bx}^{\frac{4}{3}}}$. The two firſt Terms belong to Form 24th, and the laſt to Form 25th.

Ex. 15.

Let $\frac{y^{-2}\dot{y}}{\overline{a+y}\sqrt{ab+cy-yy}}$ *be given*; Divide by $a+y$, and it will be reduced to $\frac{y^{-2}\dot{y}}{a\sqrt{ab+cy-yy}}$ $-\frac{y^{-1}\dot{y}}{aa\sqrt{ab+cy-yy}}+\frac{\dot{y}}{aa\times\overline{a+y}\sqrt{ab+cy-yy}}$; The laſt Term belongs to the 32d Form, and (by reducing the Index) the two firſt Terms become $\frac{y^{-3}\dot{y}}{a\sqrt{-1+cy^{-1}+aby^{-2}}}-\frac{y^{-2}\dot{y}}{aa\sqrt{-1+cy^{-1}+aby^{-2}}}$; both which belong to the 27th Form.

Ex. 16.

Given $\frac{d\dot{x}\sqrt{aa-bx+xx}}{a-x}$; this (by Multiplication) is reduced to $\frac{daa\dot{x}}{\overline{a-x}\sqrt{aa-bx+xx}}$ +

$dx^2\dot{x}$

$\frac{dx^2\dot{x} - bdx\dot{x}}{\overline{a - x}\sqrt{\overline{aa - bx + xx}}}$; the firſt Term belongs to Form 32. And dividing the reſt by $-x + a$, they are reduced to $\frac{-dx\dot{x}}{\sqrt{\overline{aa - bx + xx}}} + \frac{\overline{bd - ad} \times \dot{x}}{\sqrt{\overline{aa - bx + xx}}} + \frac{\overline{aad - abd} \times \dot{x}}{\overline{a - x}\sqrt{\overline{aa - bx + xx}}}$. The laſt belongs to Form 32, and the two former are Fluxions of the 27th Form.

Ex. 17.

Given $\frac{x^{-\frac{5}{3}}\dot{x} \times \overline{ab - xx}^{\frac{1}{3}}}{a^4 - 3aax^2 + x^4}$; this (by reducing the Index) becomes $\frac{x^{-5}\dot{x} \times \overline{-1 + abx^{-2}}^{\frac{1}{3}}}{1 - 3aax^{-2} + a^4x^{-4}}$, which belongs to Form 33d.

Ex. 18.

Let $\frac{z^{-1}\dot{z}\sqrt{\overline{aa - zz}}}{a^4 - aaz^2 + z^4}$ *be propoſed*; becauſe $4a^4$ is greater than a^4, it cannot be reſolved by the 33d Form, therefore I divide by $a^4 - aazz + z^4$, which reduces it to $\frac{z^{-1}\dot{z}\sqrt{\overline{aa - zz}}}{a^4} + \frac{\dot{z}\sqrt{\overline{aa - zz}}}{aa \times \overline{a^4 - a^2z^2 + z^4}} - \frac{z^3\dot{z}\sqrt{\overline{aa - zz}}}{a^4 \times \overline{a^4 - a^2z^2 + z^4}}$; the two laſt Terms belong to Form the 34th; and the firſt is further reduced to $\frac{z^{-2}\dot{z}}{aa\sqrt{\overline{1 - aaz^{-2}}}} - \frac{z\dot{z}}{a^4\sqrt{\overline{a^2 - z^2}}}$; which belong to Forms 10th and 3d.

Ex.

Ex. 19.

Let there be given $\frac{x\dot{x}}{\overline{aa-ax+xx}\sqrt{aa-ax}}$; this is reduced to $\frac{aax\dot{x}}{\overline{aa-ax+xx}\times\overline{aa-ax}|^{\frac{3}{2}}} - \frac{ax^2\dot{x}}{\overline{aa-ax+xx}\times\overline{aa-ax}|^{\frac{3}{2}}}$; the former Term belongs to Form the 35th; and the latter by Diviſion is further reduced to $\frac{-a\dot{x}}{\overline{aa-xx}|^{\frac{3}{2}}}$ + $\frac{a^3\dot{x}-a^2x\dot{x}}{\overline{aa-ax+xx}\times\overline{aa-ax}|^{\frac{3}{2}}}$; the firſt belongs to the binomial Forms, and the reſt to Form 35; where obſerve that the Part firſt found (belonging to Form 35) deſtroys a Part of this laſt Fluxion; and then the whole is reduced to this, $\frac{a^3\dot{x}}{\overline{aa-ax+xx}\times\overline{aa-ax}|^{\frac{3}{2}}}$ $-\frac{a\dot{x}}{\overline{aa-ax}|^{\frac{1}{2}}}$.

Ex. 20.

Given $\frac{z^{rn-1}\dot{z}}{a+bz^n+cz^{2n}+dz^{3n}}$, extract one Root h of the Denominator (ſuppoſed to be put $=0$), then will $h-z^n=0$, by this divide the ſaid Denominator, and let the Quotient be $e+fz^n+gz^{2n}$; then the given Fluxion is reduced to this $\frac{z^{rn-1}\dot{z}}{\overline{h-z^n}\times\overline{e+fz^n+gz^{2n}}}$, which belongs to the 29th, 30th, or 31ſt Form.

Ex. 21.

There is given $\frac{z^{rn-1}\dot{z}}{a+bz^n+cz^{2n}+dz^{3n}+tz^{4n}}$, extract two Roots a, b of the Denominator; then you have $a-z^n=0$, and $b-z^n=0$, divide the Denominator by $\overline{a-z^n}\times\overline{b-z^n}$, and let the Quotient be $e+fz^n+gz^{2n}$. Then the given Fluxion becomes $\frac{z^{rn-1}\dot{z}}{\overline{a-z^n}\times\overline{b-z^n}\times\overline{e+fz^n+gz^{2n}}} = \frac{z^{rn-1}\dot{z}}{s\times\overline{b-z^n}\times\overline{e+fz^n+gz^{2n}}} - \frac{z^{rn-1}\dot{z}}{s\times\overline{a-z^n}\times\overline{e+fz^n+gz^{2n}}}$ (putting $a-b=s$,) both which Terms belong to the 29th, 30th, or 31ſt Form.

If the given Denominator has no Roots, yet it may be divided into two Quadratics by the Rules of Algebra, as $\frac{z^{rn-1}\dot{z}}{\overline{p+qz^n+z^{2n}}\times\overline{r+sz^n+z^{2n}}}$; which may be transformed to Trinomials, as in Ex. 8. Prop. 9.

Ex. 22.

Given $\frac{z^{rn-1}\dot{z}}{\overline{a+bz^n+kz^{2n}\times e+fz^n+gz^{2n}}\Big|^m}$, here r and $2m$ muſt be Integers; put $p=\sqrt{bb-4ak}$, and the Fluxion is reſolved into theſe two Terms, $\frac{k}{p}\times\frac{z^{rn-1}\dot{z}}{\overline{\frac{b-p}{2}+kz^n}\times\overline{e+fz^n+gz^{2n}}\Big|^m} - \frac{k}{p}\times\frac{z^{rn-1}\dot{z}}{\overline{\frac{b+p}{2}+kz^n}\times\overline{e+fz^n+gz^{2n}}\Big|^m}$ then dividing by the Denominators $\frac{b-p}{2}+kz^n$, and

$\frac{b+p}{2}+kz^n$, if there be Occaſion; and the reſulting Terms will belong to the 32d Form, and perhaps to ſome others of the foregoing Forms.

Ex. 23.

To find the Fluent of $\frac{x^{\delta-1}\dot{x}}{e\pm fx^{\lambda}+x^{2\lambda}}$, where δ, λ are Integers; and $4e>ff$.

Transform it into this $x^{\delta-1}\dot{x}\times:\frac{K+kx}{1+qx+xx}+\frac{L+lx}{1+rx+xx}+$ &c. by Ex. 14. Prop. IX. and the Fluent of the ſeveral Terms will be found by Form 28.

SCHOLIUM.

Although the Conſtruction of many of the foregoing Forms depends on ſeveral Things that have not yet been delivered; yet it may be proper in this Place to give a ſhort Account thereof, that the Reader may not be intirely ignorant of them, which he will the better underſtand after he has read ſuch Parts of the following Book as they are founded on.

1. It is plain by Prop. III. that the Fluxions in Form 1, 3, 19 and 20 belong to the Fluents there aſſigned.

2. Form the 16th is calculated in Example 13, Prop. X. and Form the 18th in Example 35. Likewiſe the Fluent in Form the 15th is found by Prop. X. Aſſuming $Az^{\pi+1}+Bz^{\pi+n+1}+Cz^{\pi+2n+1}$ &c. $\times\overline{\alpha+\beta z^n}|^{\mu+1}=$ Fluent of $\overline{\alpha+\beta z^n}|^{\mu}z^{\pi}\dot{z}$.

3. By Prob. II. Sect. II. the Fluxion of any Quantity divided by that Quantity is the Fluxion of the hyperbolic Logarithm of that Quantity; and that the

Number 2.302585 reduces the common to the hyperbolic Logarithms. Now if the Fluents in Form 2, 4, 6 and 9th be put into Fluxions according to the aforeſaid Rule, it will appear that they will reſpectively produce the Fluxions in theſe Forms.

4. In Example 3d, Prob. X. Sect. II. it is ſhewn, that the hyperbolic Space between the Aſſymptotes is = Fluent of $\frac{ab}{a+x}\dot{x}$, in which if you write RR for ab, Rz (or z) for $a+x$, and R$\dot{z}$ (or $\dot{z}$) for $\dot{x}$, you will have $RR\frac{\dot{z}}{z}$, whoſe Fluent is the hyperbolic Space in Form the 2d. Likewiſe if you write RR for ab, $\frac{\alpha}{\beta}+z^n$ for $a+x$, and $nz^{n-1}\dot{z}$ for $\dot{x}$, you will have $\frac{nRR\beta z^{n-1}\dot{z}}{\alpha+\beta z^n}$ as in Form the 4th. And laſtly, write RR for ab, and $\sqrt{\alpha}+\sqrt{\beta}z^n$ for $a+x$, and $\sqrt{\beta}\times\frac{1}{2}nz^{\frac{1}{2}n-1}\dot{z}$ for $\dot{x}$; and likewiſe write $\sqrt{\alpha}-\sqrt{\beta}z^n$ for $a+x$, and $-\sqrt{\beta}\times\frac{1}{2}nz^{\frac{1}{2}n-1}\dot{z}$ for $\dot{x}$, and the Difference of theſe Fluxions will be $\frac{nRR\sqrt{\alpha\beta}\times z^{\frac{1}{2}n-1}\dot{z}}{\alpha-\beta z^n}$, as in Form the 6th.

5. By Ex. 9. Prob. X. Sect. II. the hyperbolic Sector is equal to the Fluent of $\frac{bba}{2}\times\frac{\dot{t}}{bb-tt}$, in which writing R for a and b, $R\sqrt{\frac{\beta z^n}{\alpha}}$ for t, and $\frac{n}{2}R\sqrt{\frac{\beta}{\alpha}}\times z^{\frac{n}{2}-1}\dot{z}$ for $\dot{t}$; you'll have $\frac{nRR\sqrt{\alpha\beta}\times z^{\frac{1}{2}n-1}\dot{z}}{4\times\overline{\alpha-\beta z^n}}$, as in the 6th Form.

6. And by the ſame Example, the hyperbolic Sector is $=$ Fluent of $\frac{ab\dot{y}}{2\sqrt{ab+yy}}$ $=$ (writing R for *a* and *b*) $\frac{RR\dot{y}}{2\sqrt{RR+yy}}$; now if you write $R\sqrt{\frac{\beta z^n}{\alpha}}$ for y, and $\frac{1}{2}nR\sqrt{\frac{\beta}{\alpha}}\times z^{\frac{1}{2}n-1}\dot{z}$ for $\dot{y}$, you will have $\frac{nRR\sqrt{\beta}\times z^{\frac{1}{2}n-1}\dot{z}}{4\sqrt{\alpha+\beta z^n}}$; in like manner if you put $\frac{2R}{\alpha}\sqrt{\alpha\beta z^n+\beta^2 z^{2n}}=y$, or $\frac{R}{-\alpha}\sqrt{\alpha\beta z^n+\beta^2 z^{2n}}=y$, when α is a negative Quantity, or $\frac{2R}{\sqrt{-\alpha}}\sqrt{\alpha+\beta z^n}=y$, when α is negative, and expunge y and $\dot{y}$, you will get the Fluxion of the 9th Form any of theſe Ways.

7. Since the Length of any Arch of a Circle (whoſe Radius is R) is $=\frac{3.1416R}{180}\times$ Degrees in that Arch; or Length of the Arch $=,017453R\times$ Degrees in that Arch: And by Example the 5th, Prob. VIII. Sect. II. The Arch $=$ Fluent of $\frac{RR\dot{t}}{RR+tt}$; if you write $R\sqrt{\frac{\beta z^n}{\alpha}}$ for t, and $\frac{nR}{2}\sqrt{\frac{\beta}{\alpha}}\times z^{\frac{1}{2}n-1}\dot{z}$ for $\dot{t}$, you will have $\frac{nR\sqrt{\alpha\beta}\times z^{\frac{1}{2}n-1}\dot{z}}{2\times\overline{\alpha+\beta z^n}}$, whoſe Fluent therefore is $=$ that Arch $=,017453R\times$ Degrees in the Arch $=\frac{2\ \textit{Sectors}}{R}$; from all which, the Conſtruction of Form the 5th will eaſily appear.

8. In like Manner by the ſame Problem, the Arch $=$ Fluent of $\frac{R\dot{y}}{\sqrt{RR - yy}}$, in which writing for y, $R\sqrt{\frac{\beta z^n}{\alpha}}$, or $\frac{2R}{\alpha}\sqrt{\alpha\beta z^n - \beta^2 z^{2n}}$, and the Fluxion for $\dot{y}$, you will get $\frac{nR\sqrt{\beta}}{2} \times \frac{z^{\frac{1}{2}n-1}\dot{z}}{\sqrt{\alpha - \beta z^n}}$, and $nR\sqrt{\beta} \times \frac{z^{\frac{1}{2}n-1}\dot{z}}{\sqrt{\alpha - \beta z^n}}$, from which all the other Varieties in Form the 10th are eaſily deduced.

9. Form the 11th is gradually calculated by Cor. 2. Prop. IV. where it is demonſtrated that the Fluent of $\overline{\alpha+\beta z^n}^{\mu} z^{p+n}\dot{z}$ is $= \frac{z^{p+1} \times \overline{\alpha+\beta z^n}^{\mu+1} - \overline{p+1} \times \alpha A}{\overline{\mu n+n+p+1} : \times \beta}$ in which writing π for p, φ for A, γ for $\overline{\alpha+\beta z^n}^{\mu+1}$, you will have the Fluent of $\overline{\alpha+\beta z^n}^{\mu} z^{\pi+n}\dot{z}$; again, writing $\pi+n$ for p, Fluent of $\overline{\alpha+\beta z^n}^{\mu} z^{\pi+n}\dot{z}$ (now found) for A; you will have the Fluent of $\overline{\alpha+\beta z^n}^{\mu} z^{\pi+2n}\dot{z}$; again, writing $\pi+2n$ for p, and the Fluent of $\overline{\alpha+\beta z^n}^{\mu} z^{\pi+2n}\dot{z}$ (laſt found) for A, you get the Fluent of $\overline{\alpha+\beta z^n}^{\mu} z^{\pi+3n}\dot{z}$, and ſo on.

In like Manner the 12th Form is calculated from Cor. 3. Prop. IV. where the Fluent of $\overline{\alpha+\beta z^n}^{\mu} z^{p}\dot{z} = \frac{z^{p+1}\gamma - \overline{\mu n+n+p+1} \times \beta B}{\overline{p+1} \times \alpha}$. And by a like Proceſs the 13th and 14th Forms are gradually calculated from Cor. 4 and 5, Prop. IV. Theſe four Forms might have been differently expreſs'd from what they are in the Table. For Example, Form the 11th may be expreſſed thus; let $e = \pi + 1$, $d = \pi + 1 + n + \mu n$, then the Fluent of $\overline{\alpha + \beta z^{n}}|^{\mu} z^{\pi + \lambda n}\dot{z} =$

$$\frac{\overline{-\alpha}|^{\lambda} \times e \times \overline{e+n} \times \overline{e+2n} \times \overline{e+3n} \times \&c.\text{ to } \lambda \text{ Places}}{\beta^{\lambda} \times d \times \overline{d+n} \times \overline{d+2n} \times \overline{d+3n} \times \&c.\text{ to } \lambda \text{ Places}} \times \varphi$$

$$-\frac{z^{e}\gamma}{e\alpha\varphi}\text{A} - \frac{d\beta z^{n}}{\overline{e+n}\cdot\alpha}\text{B} - \frac{\overline{d+n}\cdot\beta z^{n}}{\overline{e+2n}\cdot\alpha}\text{C} - \frac{\overline{d+2n}\cdot\beta z^{n}}{\overline{e+3n}\cdot\alpha}\text{D} - \&c.$$

continued to $\lambda + 1$ Places; where A, B, C, *&c.* are the firſt, ſecond, third, *&c.* Terms with their Signs, $\varphi =$ Fluent of $\overline{\alpha + \beta z^{n}}|^{\mu} z^{\pi}\dot{z}$, $\gamma = \overline{\alpha + \beta z^{n}}|^{\mu + 1}$.

10. The 17th Form is derived from the 11th; for when z and γ are 0, all the Terms vaniſh except the firſt, in which taking 0, 1, 2, 3, *&c.* Terms (for each of the Terms in the Quantity $e + fz^{n} + gz^{2n}$ *&c.*) and multiplying each by its reſpective Coefficient, and the Sum of all by φ, you will at laſt obtain this Form.

11. The Fluxions in the 21ſt and following Forms are transform'd by Prop. IX. and their Truth will appear by the bare Subſtitution of the Quantities therein contained.

12. There ſtill remain the 7th and 8th Forms, whoſe Calculus is ſomething more difficult: Theſe I inveſtigate after the following manner. Let T, U, W, *&c.* ſtand for the Quantities $\sqrt{1 - 2sx + xx}$, $\sqrt{1 - 2tx + xx}$, $\sqrt{1 - 2ux + xx}$, *&c.* then the Fluxion of L × Log. T (or the hyperbolic Log. T) will be $\frac{-s\dot{x} + x\dot{x}}{1 - 2sx + xx}$, and the like for U, W, *&c.* as s plain from what is delivered in Art. 3. Again, he Fluxion of NP (as will appear by Art. 8.) is $\frac{a\dot{x}}{-2sx + xx}$, and ſo for the reſt NQ, NR, *&c.*

Laſtly, we muſt take for granted (for I ſhall not ſtay to demonſtrate it here) that the Product of all the Squares of T, U, W, *&c.* if λ is an even Number, or the laſt drawn into the Squares of all the reſt, if λ is an odd Number, is always equal to $1+x^{\lambda}$. Theſe Things premiſed, let us inveſtigate any one Caſe; as for Example, to find the Fluent of $\frac{\dot{x}}{1+x^3}$; here $\lambda=3$ and $\frac{180}{3}=60$: Here b, c, u, *&c.* $=0$, and $t=-1$, and therefore $\sqrt{1-2tx+xx}=1+x$; and all Arches above 180 are excluded; hence I aſſume $LA\times\text{Log.}\sqrt{1-2sx+xx}+BNP+CL\times\text{Log.}\overline{1+x}=$ Fluent of $\frac{\dot{x}}{1+x^3}$. This in Fluxions gives

$$\frac{Ba-As+Ax}{1-2sx+xx}\dot{x}+\frac{C\dot{x}}{1+x}=\frac{\dot{x}}{1+x^3};$$ put $Ba-As=\mathrm{I}$.

then $$\frac{\mathrm{I}+Ax}{1-2sx+xx}+\frac{C}{1+x}=\frac{1}{1+x^3};$$ reduce them to a common Denominator, then

$$\frac{\begin{array}{llll} +\mathrm{I} & +Ax & +Ax^2 & \\ +C & \begin{array}{l}+\mathrm{I}\\-2sC\end{array} & +C & \end{array}}{\begin{array}{llll} 1 & -2sx & +1 & \\ & +1 & -2s & \end{array}x^2+x^3}=\frac{1}{1+x^3}.$$

Here the Denominators being equal, we have $1-2s=0$, or $2s=1$; and equating the Coefficients of the homologous Terms in the Numerator, $\mathrm{I}+C=1$, $A+\mathrm{I}-2sC$ or $A+\mathrm{I}-C=0$, and $A+C=0$. From whence will be found $A=-\frac{1}{3}$, $C=\frac{1}{3}$, $\mathrm{I}=\frac{2}{3}$, $B=\frac{1}{2a}=$ (becauſe $aa=\frac{3}{4}$,) $\frac{2}{3}a$. Whence the Fluent is $-\frac{1}{3}L\times\text{Log.}\sqrt{1-2sx+xx}+\frac{2}{3}aNP+\frac{1}{3}L\times\text{Log.}\overline{1+x}$.

This done, I put $x^{\lambda} = \frac{\beta z^{n}}{\alpha}$; and (by Prop. IX.) transform the Fluxion $\frac{\dot{x}}{1+x^{3}}$ into this other $\frac{n\alpha^{\frac{2}{3}}\beta^{\frac{1}{3}}}{3} \times \frac{z^{\frac{1}{3}n-1}\dot{z}}{\alpha+\beta z^{n}}$, whoſe Fluent therefore you have above.

In equating the Ccefficients, if you had made $A+I-C=1$, and the other Equations $=0$, you would have got the Fluent of $\frac{x\dot{x}}{1+x^{3}}$.

After the ſame Manner may the 8th Form be inveſtigated, remembring what was ſaid of Sines and Logarithms, and obſerving that (if you put T, U, W, &c. for $1-x$, $\sqrt{1-2sx+xx}$, $\sqrt{1-2tx+xx}$), the firſt T into the Squares of all the reſt, if λ is an odd Nnmber; or the firſt and laſt into the Squares of the reſt, if λ is an even Number, is always equal to $1-x^{\lambda}$: The Demonſtration of which belongs not to this Place.

Here note that in theſe two Forms, you may if you pleaſe put P, Q, &c. for the Degrees of Arches whoſe Sines are $\frac{a}{\sqrt{1-2sx+xx}}$, $\frac{b}{\sqrt{1-2tx+xx}}$ &c. their Fluxions being the ſame as the other. But then if the Arch whoſe Sine is $\frac{ax}{\sqrt{1-2sx+xx}}$ be leſs than a Quadrant, the Arch whoſe Sine is $\frac{a}{\sqrt{1-2sx+xx}}$ will be greater than a Quadrant.

PROP. XII.

To correct the Fluent of a given Fluxionary Equation or Proportion.

When the Fluent is obtained, by either of the foregoing Propoſitions, from any given fluxionary Equation or Proportion; it is only obtained in general. But ſince the Deſign in any particular Problem is to find the contemporary Fluents; ſuch general Equation or Proportion therefore is for the moſt part imperfect till it be corrected by the following

RULE.

1. Inſtead of each ſeveral variable Quantity (or Member) in the Fluent *ſubſtitute* ſuch a *determinate Value* thereof, as each of them is known to have in any one certain and *particular Point* or Place: Then *ſubſtract* each *Side* of the reſulting Equation from the *correſponding Side* of the *Fluent*; and the remaining Equation will be the *correct Fluent*. And the ſame Rule obtains when the Fluent is expreſſed by a general Proportion.

2. Or any particular Part of the Fluent may be otherwiſe had thus without the foregoing Correction: *Subſtitute* the Values of the variable Quantities for any particular *Time*, Place or Point; do the ſame for another given Time or Place; and the *Difference* of the reſulting Equations, gives the correſponding Part of the *Fluent*.

3. Sometimes it may be ſufficient to add ſome *given Quantity* on one Side of the Equation, which may afterwards be determin'd according to the Nature and Circumſtances of the Queſtion.

DEMON-

DEMONSTRATION.

Let $X=Z$ be the Fluent obtained in general, from a given fluxionary Equation, $\dot{X}=\dot{Z}$. Now ſince X may not be equal to Z (by Cor. 2. Prop. II.), take d a given Quantity and let $X=Z+d$ be an Equation for the contemporary Fluents. Now at a certain Time when $X=b$, let $Z=c$; then you will have the particular Equation $b=c+d$ at that Time; this Equation taken from the former will leave this general Equation for the contemporary Fluents, $X-b=Z+d-c-d$, that is $X-b=Z-c$. Q. E. D.

SCHOLIUM.

Theſe Things are to be thus underſtood when the variable Quantity in the Fluent continually increaſes (or decreaſes). But in Caſe it increaſes and decreaſes by Turns, or paſſes through one or more *Maximums* or *Minimums*: Then the ſeveral Parts of the Fluent, between any given Point and each Maximum or Minimum muſt be *ſeparately* found by diſtinct Operations, and each *corrected* by this Propoſition, and then the ſeveral Parts *collected* into one *Sum.*

EXAMPLE I.

Let $a\dot{x}=2y\dot{y}$, and the Fluent is $ax=yy$, now when $y=0$, let $x=0$, and then $ax-0=y^2-0$, or $ax=yy$, which therefore needs no Correction.

Ex. 2.

Again let $a\dot{x}=2y\dot{y}$, and finding the Fluent $ax=yy$. Now when $y=0$, let $x=b$; and the Equation becomes $ab=0$, this ſubſtracted from the other Equation, leaves $ax-ab=yy$, the contemporary Fluent required.

Ex. 3.

Let $b\dot{x} - x\dot{x} = y\dot{y}$, the Fluent is $bx - \frac{x^2}{2} = \frac{y^2}{2}$, but at a certain Point of Time $x = b$, and alſo $y = r$, then the Equation becomes $bb - \frac{1}{2}bb$ or $\frac{1}{2}bb = \frac{rr}{2}$, therefore the correct Fluent is $bx - \frac{1}{2}xx - \frac{1}{2}bb = \frac{1}{2}yy - \frac{1}{2}rr$. That is by Reduction $rr - bb + 2bx - xx = yy$.

Ex. 4.

Suppoſe $2by\dot{y} = \frac{3cx^2\dot{x}}{z} - \frac{cx^3\dot{z}}{z^2}$, the Fluent is $byy = \frac{cx^3}{z}$; but when $y = r$, the Quantity $\frac{cx^3}{z} = A$. Therefore the correct Fluent is $b \times \overline{y^2 - r^2} = \frac{cx^3}{z} - A$.

Ex. 5.

Let $\frac{\dot{y}^2}{2b} = \frac{\dot{x}\ddot{x}}{\sqrt{\dot{y}^2 + \dot{x}^2}}$, *ſuppoſing* $\dot{y}$ *given*; the Fluent is $\frac{y\dot{y}}{2b} = \sqrt{\dot{x}^2 + \dot{y}^2}$, but when $y = a$, $\dot{x} = 0$, and $\dot{y} = \sqrt{\dot{y}^2 + \dot{x}^2}$, therefore the contemporary Fluent is $\frac{y\dot{y} - a\dot{y}}{2b} = \sqrt{\dot{x}^2 + \dot{y}^2} - \dot{y}$, or $\overline{y - a + 2b} \times \dot{y} = 2b\sqrt{\dot{x}^2 + \dot{y}^2}$.

Ex. 6.

Let $b\dot{y} = \frac{\dot{x}}{\sqrt{aa + xx}}$: By Form the 9th its Fluent is $by = 2.3025 \text{ Log. } x + \sqrt{aa + xx}$; but when $y = a$, $x = 0$, then the Equation is $ba = 2.3025 \text{ Log. } a$. Therefore the corrected Fluent is $by - ba = 2.3025 \text{ Log. } x + \sqrt{aa + xx} : - 2.3025$

Log. a; that is $by - ba = 2.302585$ Log. $\frac{x + \sqrt{aa + xx}}{a}$.

PROP. XIII.

To inveſtigate a Problem by the Method of Fluxions.

RULES.

1. Let all the Quantities be denoted by proper Symbols, as is explained in the Notation of Fluxions, and let ſome *one* of the variable Quantities (with which the others may always be compared) be ſuppoſed to increaſe uniformly: And this may be called the *Principal* variable Quantity. Then the given Equations, or ſuch Equations as are deduced from the Conditions of the Problem, muſt be turn'd into Fluxions, ſecond Fluxions, *&c.* by Prop. III. in order to get as many Equations among theſe Fluxions, as you have Occaſion for.

2. But becauſe ſometimes ſome Doubt may ariſe about the Signs of the Fluxions: Obſerve that any fluxionary Equations, deduced from the Equations of Curves, or from any given Equations in the Problem, will contain the Fluxions with their proper Signs. But in any Proportions made between Fluxions or Moments, as in ſimilar fluxionary Triangles and the like; then the Fluxions or Moments of Quantities that decreaſe muſt be actually made *negative*; and thoſe that increaſe muſt be written *affirmative:* Or, which is the ſame Thing, (ſince one Part of any Whole increaſes whilſt the other Part of it decreaſes, therefore) inſtead of the negative Fluxion, you may take the proper Fluxion of the increaſing Part of that Quantity.

3. Since *Velocity* is always meaſured by the *Spaces* uniformly deſcribed thereby; ſo may the *Fluxions* be meaſured by the *Moments* uniformly generated by the Fluxions. Therefore the Moments (uniformly generated or, which amounts to the ſame Thing, conſidered as *ariſing* or *vaniſhing*) may be put for the Fluxions, and the Reſult will always be the very ſame in all Operations. And ſince in many Caſes, eſpecially in geometrical Problems, the reaſoning and calculating with the *Moments* will be more eaſy and evident than with the *Fluxions*; the Equations that are gained thereby muſt at laſt be changed into fluxionary Equations, by ſubſtituting the Fluxions inſtead of the Moments, which muſt always be ſuppoſed to be taken in the *firſt Inſtant* of their Generation: Or, *at leaſt* when the Operation is over, theſe *Moments* muſt be ſuppoſed to be *diminiſh'd ad infinitum*, that their *firſt Ratio* may be always obtained.

4. In the Reſolution of any *Problem*, the Nature and Conditions of it are to be cloſely examin'd and ſtrictly purſued according to all the known Methods of *Algebraic Reaſoning*, by attentively conſidering the *Relations* of the Quantities, and their mutual *Proportions* and *Dependance* on one another; and forming your Proceſs according to theſe their Properties, by duly comparing together the *Quantities*, their *Moments* or *Increments*, their *Fluxions* or *ſecond Fluxions*, &c. as the Caſe requires; till you get a competent Number of *Equations* or *general Proportions*. And then you muſt proceed to expunge ſuch Quantities as are ſuperfluous, till at laſt you get a fluxionary *Equation* or *Proportion* with the Quantities required. Then if there be Occaſion,

5. Find the *Fluent* of the ſaid Equation or general Proportion by Prop. X. or XI. and correct it by Prop. XII. And then you have a *compleat Equation* or *general Proportion* containing the Quantities ſought.

6. But to obtain an *Equation* of the indetermin'd Quantities, by having the correct general *Proportion* before found, or by having only the fluxionary Proportion; you muſt aſſign to each indetermined Quantity in the ſaid Proportion, (or in the Fluxion thereof) ſuch a determined Value as it is known to have in any particular Caſe; and from thence you muſt draw an Analogy from the *Fluxion* alone (of the general Proportion), or from the *correct Fluent* alone, (or ſometimes from both together.) From whence there will be had an *Equation* between the Quantities required; or at leaſt between their Fluxions, whoſe *Fluent* muſt then be *found* and *corrected*. And note, theſe *determin'd Values* (of the Quantities) may be either expreſs'd in Numbers or Symbols, as any one ſhall think proper.

Sometimes it may be ſufficient to aſſume a *given Quantity*, by which multiplying one Side of the Proportion it will be turned into an *Equation*; and this given Quantity may afterwards be determined according to the Nature of the Queſtion.

Theſe are the *general Rules*, but after all, many Things muſt be left to the Sagacity and Invention of he Artiſt.

Corol. Hence every *Problem* belongs to *Fluxions*, n which the *Increments*, or the *Proportions* of the Inrements or Moments of the ſeveral variable Quanities contained therein, can in *all Caſes* be computed nd expreſſed by *Equations*.

Example I.

To find the Velocity wherewith the Ordinate BM of a Circle increaſes in every Point, whilſt it moves uniformly along the Diameter AD. FIG. 1.

Let $AD=2r$, $AB=x$, $BM=y$; and by the Naure of the Circle $2rx-xx=yy$; this Equation put into

FIG. 1. into Fluxions gives $2r\dot{x}-2x\dot{x}=2y\dot{y}$, or $\frac{r-x}{y}\dot{x}=\dot{y}$, general Equation for the Increaſe of y in all Points. Therefore in *A* where y is o, and x is o, $\frac{r-x}{y}\dot{x}=\dot{y}$ becomes $\frac{r\dot{x}}{0}=\dot{y}$, therefore $\dot{y}$ is infinite. If $CB=BM$, or $r-x=y$, then $\dot{x}=\dot{y}$, and x and y increaſe equally. But in *C* where $r-x=0$, then $\dot{y}-0$, therefore y does not increaſe at all. In *b* where $Cb=bm$, then $\dot{y}=-\dot{x}$, therefore y decreaſes as faſt as x increaſes. Laſtly in *D* where $y=0$, and $x=2r$, $\dot{y}=\frac{-r\dot{x}}{0}=-$ Infinity, and there the Ordinate decreaſes infinitely: And in all the intermediate Points it has all the intermediate Degrees of Increaſe or Decreaſe.

Ex. 2.

To find the Space a deſcending Body will deſcribe in any Time by the uniform Force of Gravity.

Let $z=$ Space, $x=$ Time, $v=$ Velocity, acquired in that Time. By the Principles of Mechanics $z' \propto vx' \propto xx'$, and therefore $\dot{z}\propto v\dot{x}\propto x\dot{x}$; that is the Fluxion of the Space is every where as the Velocity into the Fluxion of the Time; that is, (becauſe the Velocity is as the Time) as the Time into the Fluxion of the Time.

Now if only the *Ratio* of the Space and Time be required, it will be ſufficient to take the Fluent; and then $z\propto\frac{x^2}{2}$, or $z\propto x^2$, that is, the Space is always as the Square of the Time.

But if the abſolute Quantity of the Space is ſought, we muſt reduce the general Proportion or its Fluxion to an Analogy from ſome particular known Caſe.

Thus it is known by Experiment that in the Time t or 1 Second, a Body would acquire ſuch a Velocity as to move through s or $32\frac{1}{6}$ Feet uniformly in that Time, or to have deſcended through $\frac{1}{2}s$ or $16\frac{1}{12}$ Feet in that Time. Whence FIG. 1.

$$t : s :: \dot{x} : \frac{s\dot{x}}{t} = \text{Fluxion of the Space } z \text{ when } x = t;$$

wherefore from the general Analogy ($\dot{z} \propto x\dot{x}$) we have $\frac{s\dot{x}}{t} : t\dot{x} :: \dot{z} : x\dot{x}$, and $\dot{z} = \frac{sx\dot{x}}{tt}$, and finding the Fluent, $z = \frac{sx^2}{2tt}$, which needs no Correction (becauſe when $z=0$, $x=0$).

Or thus from the Fluxion and Fluent, $\frac{s\dot{x}}{t} : t\dot{x} :: z : \frac{x^2}{2}$, whence $z = \frac{sxx}{2tt}$.

Or laſtly thus, ſince $\frac{1}{2}s = z$, when $t = x$, therefore from the general Analogy ($z \propto xx$) it will be $\frac{1}{2}s : tt :: z : xx$; whence $z = \frac{sxx}{2tt}$, and thus the ſame Equation is obtained any of theſe Ways.

Ex. 3.

If a Body is projected upwards with a given Velocity a, *to find how far it will aſcend in any Time* x.

Let z = Space, v = its Velocity; then by Mechanics $\acute{z} \propto v\acute{x}$. Now ſince the firſt Velocity is given, therefore the Space s, which would be uniformly deſcribed in the Time t, of its whole Aſcent, will be given; wherefore $t : s :: \acute{x} : \frac{s\acute{x}}{t}$ = Moment of the Space $\acute{z}$ at the firſt Inſtant, that is when $a = v$.

There-

FIG. 1. Therefore from the general Proportion ($\acute{z} \propto v\acute{x}$,) $\frac{s\acute{x}}{t} : a\acute{x} :: \dot{z} : v\acute{x} :: \dot{z} : v\dot{x}$; whence $\dot{z} = \frac{sv\dot{x}}{ta}$; but the Velocity the Body loſes is as the Time, therefore $t : a :: x : \frac{ax}{t} =$ Velocity loſt, whence $v = a - \frac{ax}{t}$, therefore $\dot{z} = \frac{s\dot{x}}{t} - \frac{sx\dot{x}}{tt}$; and the Fluent $z = \frac{sx}{t} - \frac{sx^2}{2tt}$, which needs no Correction.

Hence if x be greater than $2t$, the Body will have deſcended again below the Point it was projected from.

Otherwiſe thus.

Let $z =$ Space, $v =$ Velocity, $s =$ Velocity generated by Gravity in the Time t; then by *Mechanics* $\acute{z} \propto v\acute{x}$. Now ſince the firſt Velocity is given; therefore the Space d, which would thereby be uniformly deſcribed in a given Time t, will be given; wherefore $t : d :: \acute{x} : \frac{d\acute{x}}{t} =$ Moment of the Space $\acute{z}$ at the firſt Inſtant, that is when $a = v$. Therefore from the general Proportion ($\acute{z} \propto v\acute{x}$) $\frac{d\acute{x}}{t} : a\acute{x} :: \acute{z} : v\acute{x} :: \dot{z} : v\dot{x}$, whence $\dot{z} = \frac{dv\dot{x}}{ta}$; but the Velocity the Body loſes is as the Time; therefore $t : s :: x : \frac{sx}{t} =$ Vel. loſt; whence $v = a - \frac{sx}{t}$, therefore $\dot{z} = \frac{d\dot{x}}{t} - \frac{dsx\dot{x}}{att}$; and the Fluent $z =$

$\frac{dx}{t} - \frac{dsxx}{2att}$; but $t : d :: 1 : a = \frac{d}{t}$; there- FIG. fore $z = ax - \frac{sxx}{2t}$.

Hence if $2ta$ be greater than sx, the Body will have deſcended again below the Point it was projected from.

Ex. 4.

To find the Time wherein a given Cylinder full of Water will empty itſelf by a Hole at the Bottom.

Let $AC = b$, $CE = x$, $AE = b - x$, $t =$ Time of running out with the firſt Velocity, $z =$ Time ſought. 2.

Now the Moment of the Quantity running out $\propto \dot{z}$ $\times$ Velocity; but the Velocity is as $\sqrt{CE}$, and the Moment of the Quantity is as the Moment of AE, or $-\dot{x}$, whence $-\dot{x} \propto \dot{z}\sqrt{x}$, therefore $\dot{z} \propto \frac{-\dot{x}}{\sqrt{x}}$; But $b : t :: -\dot{x} : \frac{-t\dot{x}}{b} =$ Moment of the Time at the firſt Inſtant. Therefore (from the general Proportion $\dot{z} \propto \frac{-\dot{x}}{\sqrt{x}}$), $\frac{-t\dot{x}}{b} : \frac{-\dot{x}}{\sqrt{b}} :: \dot{z} : \frac{-\dot{x}}{\sqrt{x}}$: $\dot{z} : \frac{-\dot{x}}{\sqrt{x}}$, whence $\dot{z} = \frac{-t\dot{x}}{\sqrt{bx}}$, and the Fluent is $z = \frac{-2tx^{\frac{1}{2}}}{\sqrt{b}}$; but in the Point A, $z = 0$, and $x = b$, therefore the Fluent corrected (by Prop. XII.) is $z = 2t - \frac{2tx^{\frac{1}{2}}}{\sqrt{b}}$; and when $x = 0$, then the whole Time $z = 2t$.

EX. 5.

FIG. 3. *To find the Time wherein a given Fruſtum of a Cone, full of Water will empty itſelf by a Hole in the Bottom.*

Let the Cone be compleated, and put $VG = p$ the Altitude; $TG = h$ the Height of the Fruſtum, $TS = x$, Circle $CD = d$, $VT = b$, $t =$ Time of running out (of a Cylinder whoſe Baſe is d and Height h) with the firſt or greateſt Velocity, $z =$ Time ſought. Proceeding here as in the laſt Example, you'll find $\acute{z}\sqrt{x} \propto$ Moment of the Quantity $\propto -\acute{x} \times$ Circle $EF \propto \frac{-d\acute{x}}{pp} \times \overline{b+x}^{2}$. Therefore $\acute{z} \propto \frac{-d\acute{x}}{pp\sqrt{x}} \times \overline{b+x}^{2}$; then $\frac{-t\acute{x}}{h} : \frac{-d\acute{x}}{\sqrt{h}} :: \acute{z} : \frac{-d\acute{x}}{pp\sqrt{x}} \times \overline{b+x}^{2} :: \dot{z} : \frac{-d\dot{x}}{pp\sqrt{x}} \times \overline{b+x}^{2}$, from the general Analogy; therefore $\dot{z} = \frac{-t\dot{x}}{pp\sqrt{hx}} \times \overline{b+x}^{2}$. And the Fluent is $z = \frac{-t}{pp\sqrt{h}} \times : 2bbx^{\frac{1}{2}} + \frac{4}{3}bx^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}}$; and when corrected, the whole Time $z = t \times \frac{2bb + {}^{4}bh + \frac{2}{5}hh}{pp}$.

And if the Fruſtum were inverted, the Time would be found to be $t \times \frac{2pp - \frac{4}{3}ph + \frac{2}{5}hh}{bb}$. Where $d =$ Circle AB, $x = GS$.

SCHOLIUM.

If EF or y be the double Ordinate in any Curve CA; the Time of running out might have been found the ſame Way, only by ſubſtituting the Value of EF

or y had from the Nature of the Curve, into the FIG. Equation $\dot{z} = \frac{-ty^2\dot{x}}{d\sqrt{hx}}$, and then finding the Fluent, where $d =$ Square of the Diameter of any Cylinder, and $h =$ Height, $t =$ Time of the Cylinders running out with that firſt Velocity.

Ex. 6.

Let ACE be (the Section of) a Wall ſupporting a Fluid behind it, and joining to the perpendicular Side AC; to find the Curve ADE terminating the other Side of the Wall, ſo that its Strength may be every where as the Preſſure it ſuſtains. 4.

Let $AC = h$, $AB = x$, $BD = y$. The Effect which any Number of Particles of the Fluid preſſing at B have to break the Wall at C, is as $CB \times$ Number of Particles × their Force, that is $\propto \overline{h-x} \times \dot{x} \times x$, (becauſe the Number is as $\dot{x}$, and the Preſſure or Force as x). And the Sum of all the Forces acting on AB to break it at C is as the Sum of all the $\overline{h-x} \times x\dot{x}$ that is as the Fluent of $hx\dot{x} - x^2\dot{x}$, and therefore as $\frac{hx^2}{2} - \frac{x^3}{3}$, and when $x = h$, the whole Preſſure on AC to break it at C will be $\frac{1}{6}h^3$; therefore the Effects of the Preſſure at B and C will be as AB^3 and AC^3. But the Strength of the Wall in B and C is ſuppoſed to be as theſe Forces, and by Mechanics 'tis known to be as BD^2 and CE^2; Therefore $AB^3 : AC^3 :: BD^2 : CE^2$, that is $y^2 \propto x^3$; And the Curve ADE is a Semicubical Parabola whoſe Vertex is A, and therefore convex towards AC.

Ex. 7.

FIG. 4. *Suppoſe a Wind to blow againſt the perpendicular Side AC of the Wall ACE; to find the Curve ADE bounding the other Side, ſo as the Strength of it be every where as the Force it ſuſtains.*

Let $AB=x$, $BD=y$, $AC=h$. The Force of any Particles at B to break the Wall at C is as $CB\times$ Number of Particles $\propto \overline{h-x}\times\dot{x}$ or $\propto\overline{h-x}\times\dot{x}$; and therefore the whole Force of all the Particles on AB to break the Wall at C is $\propto$ the Fluent of $h\dot{x}-x\dot{x}$ $\propto hx-\frac{x^2}{2}$; therefore the Force to break it at C by all the Particles on AC is $\frac{1}{2}hh$ or as hh, and this muſt be as the Strength of the Wall or as CE^2; conſequently $x^2\propto y^2$ and $x\propto y$, therefore ADE is a right Line.

Ex. 8.

5. *Let ABC be a heavy Body, BC a Spring fixt to the Block D: Let AB be cloſe thruſt up to C, that the Spring may be cloſe bent, and fixt thus to the Stock D by a Pin. To find with what Velocity the Body will be projected by the Force of the Spring when the Pin is ſuddenly pluck'd out.*

Let $BC=b$ the Length of the Spring in its natural Poſition, $CN=x$, $v=$ Velocity of the Point B when it arrives at N, $w=$ Weight of the Body AB, $t=$ Time of deſcribing CN. By Mechanics or the Laws of Motion $\dot{v}\propto\frac{force\times\dot{t}}{Body}\propto$ (by the Nature of a Spring) $\frac{b-x}{w}\dot{t}$: Likewiſe by the Laws of Motion $\dot{t}\propto\frac{\dot{x}}{v}$; therefore $\dot{v}\propto\frac{b-x}{w}\times\frac{\dot{x}}{v}$ or $v\dot{v}\propto\frac{b\dot{x}-x\dot{x}}{w}$, and

finding the Fluent, $v^2 \propto \frac{2bx - xx}{w}$, and when $x = b$, FIG.
$v^2 \propto \frac{bb}{w}$; that is the Square of the Velocity is reciprocally as the Weight of the Body: Conſequently, if the Body is projected horizontally, the Square of the Diſtance it is projected to will alſo be reciprocally as the Body.

Univerſally, ſince $b - x$ repreſents the Force of the Spring at N; and b the Force of it at C; if inſtead of b, we put $f =$ abſolute Force of the Spring at C, then will $\frac{bf}{w}$ be as the Square of the Velocity for all Springs.

Ex. 9.

Let BC be the Quadrant of a Circle, A the Center, RS parallel to BA. To find the Nature of the Curve DFQ that conſtantly biſſects the Angle made by RF and the Arch FC. 6.

Deſcribe the Circle ne infinitely near FC, and draw npA, and no parallel to CA; then ſince the Angle $oFn = nFp$, and Side Fn common, and the Angles o, p right, therefore $on = np$.

Let $AS = x$, $SF = y$, AB or $Ap = v$, $AD = b$, and ſince $on = np$, that is $\acute{x} = \acute{v}$, therefore $\dot{x} = \dot{v}$, and the Fluent $x = v$, and corrected $x = v - b$, or $b + x = v$. But by Prop. 47. *Eu.* I. $\overline{b+x}^2$ $(= v^2)$ $= yy + xx$, that is $bb + 2bx = yy$; whence the Curve DFQ is a Parabola, whoſe Latus rectum is $2b$, and Focus the Point A.

EX. 10.

FIG. 7. *To find the Time of the Vibration of a Pendulum in an extreamly ſmall Arch of a Circle; or in any other Arch.*

Let the Length of the Pendulum $CB = r$, Cord $AB = c$, $BF = x$, Arch $BE = v$, $Ee = \dot{v}$, $Ff = \dot{x}$, then $BD = \frac{cc}{2r}$, $BG = \frac{cx}{2r}$, $DG = \frac{c}{2r} \times \overline{c - x}$, $EG = \sqrt{cx - \frac{ccxx}{4rr}}$. Alſo let t = Time of a Body's deſcending or aſcending thro' the Cord AB, z = Time of deſcending or aſcending thro' the Arch BE, then $\frac{1}{2}t$ = Time of deſcribing AB with the Velocity in B; by the Nature of the Circle $\dot{v} = \dfrac{c\dot{x}}{2\sqrt{cx - \frac{ccxx}{4rr}}}$. The Times of deſcribing any Spaces uniformly are as the Spaces directly and the Velocities reciprocally; but ſince the Pendulum falls from A and is ſuppoſed to deſcribe the Arch AB in deſcending, or BA in aſcending, therefore the Velocities in B and E are as $\sqrt{DB}$ and $\sqrt{DG}$, or as $\sqrt{AB}$ and $\sqrt{AF}$; therefore $\frac{1}{2}t : \frac{c}{\sqrt{c}} :: \dot{z} : \frac{\dot{v}}{\sqrt{c-x}}$ $:: \dot{z} : \frac{\dot{v}}{\sqrt{c-x}}$; whence $\dot{z} = \dfrac{t\dot{v}}{2\sqrt{cc - cx}} = \dfrac{trx^{-\frac{1}{2}}\dot{x}}{2\sqrt{4crr \begin{matrix} -4rr \\ -cc \end{matrix} x + cxx}}$ = (rejecting $cxx - ccx$ as extreamly ſmall) $\dfrac{tx^{-\frac{1}{2}}\dot{x}}{4\sqrt{c - x}}$: And the Fluent (by Form 10th) is $z = \frac{t}{2r} \times$ Arch of this Circle FIG. 7. whoſe Sine is $r\sqrt{\frac{x}{c}}$, and when $x = c$, then $z = \frac{t}{2r} \times$ Quadrant $BH = \frac{t}{4} \times 3.1416$: And $2z$ or the Time of an entire Vibration is $= \frac{1}{2}t \times 3.1416$; Or, which is the ſame Thing, $2z = \frac{3.1416}{2} \times$ Time of deſcending through twice the Length of the Pendulum. Or putting $s = 16\frac{1}{12}$ Feet, $r =$ Feet in the Pendulum, then $2z = \frac{3.1416}{2} \times \sqrt{\frac{2r}{s}}$, in Seconds.

And to find the Time of Vibration in any Arch AB; we have $\dot{z} = \frac{trx^{-\frac{1}{2}}\dot{x}}{2\sqrt{4crr - 4r^2x - ccx + cxx}} = \frac{tx^{-\frac{1}{2}}\dot{x}}{4\sqrt{c-x}\sqrt{1-\frac{cx}{4rr}}} = \frac{tx^{-\frac{1}{2}}\dot{x}}{4\sqrt{c-x}} \times \overline{1-\frac{cx}{4rr}}^{-\frac{1}{2}}$ $= \frac{tx^{-\frac{1}{2}}\dot{x}}{4\sqrt{c-x}} \times \overline{1-dx}^{-\frac{1}{2}}$ (putting $d = \frac{c}{4rr}$) $= \frac{tx^{-\frac{1}{2}}\dot{x}}{4\sqrt{c-\dot{x}}} \times : 1 + \frac{1}{2}dx + \frac{1.3}{2.4}d^2x^2 + \frac{1.3.5}{2.4.6}d^3x^3$ &c. whoſe Fluent, by Form 17th, is $\varphi \times : 1 + \frac{cd}{2^2} + \frac{3^2}{2^2.4^2}c^2d^2 + \frac{3^2.5^2}{2^2 4^2.6^2}c^3d^3$ &c. $= \varphi \times : 1 + \frac{cc}{2^2.4rr}A + \frac{3^2cc}{4^2.4rr}B + \frac{5^2cc}{6^2.4rr}C$. Where $\varphi = F : \frac{tx^{-\frac{1}{2}}\dot{x}}{4\sqrt{c-x}} = \frac{3.1416t}{4}$, the Time in a very ſmall Arch.

FIG. 7. Therefore the Time of Vibration in a very ſmall Arch, is to the Time in the Arch, the Chord of whoſe Half is c, as 1 to $1 + \frac{cc}{2^2.4rr}A + \frac{3^2cc}{4^2.4rr}B + \frac{5^2cc}{6^2.4rr}C + \frac{7^2cc}{8^2.4rr}D +$ &c.

COR. Hence if the *Pendulum* r meaſures *Time* by *vibrating* in a very *ſmall Arch*; and if c be the *Chord* of the *Arch* a; then the *Seconds* loſt in 24 *Hours* by vibrating in the Arch $2a$, will be nearly $\frac{86400 \times cc}{16rr}$ or $\frac{5400cc}{rr}$, and the Minutes loſt $\frac{90cc}{rr}$.

Or if *Time* be meaſured by vibrating in the *Arch* $2a$, then the *Seconds* loſt by vibrating in the Arch $2A$ (C being the Chord of A) will be $\frac{5400}{rr} \times \overline{CC - cc}$, nearly.

EX. II.

8. *To find the Meridional Parts for any Latitude.*

Let Radius $CA = r$, the given Arch of Latitude $AB = v$, Sine $DB = y$, Meridional Parts of $AB = z$. By Conſtruction of *Mercator*'s Chart, as Coſine of the Latitude $(\sqrt{rr - yy})$: Radius (r) : : $(\dot{v} : \dot{z} : :)$ $\dot{v} : \dot{z} = \frac{r\dot{v}}{\sqrt{rr - yy}}$, but by the Nature of the Circle $\dot{v} = \frac{r\dot{y}}{\sqrt{rr - yy}}$; whence $\dot{z} = \frac{rr\dot{y}}{rr - yy}$; whence, by Form the 6th, $z = \frac{2.30258r}{2} \times \text{Log.} \frac{r + y}{r - y} = 2.30258r \times \text{Log.} \sqrt{\frac{r + y}{r - y}}$. But in the

Triangle EBF, as EB $(\sqrt{rr-yy})$: Rad. (r) : : EF $(r+y)$: Tangent of the Angle $B = \frac{r\times\overline{r+y}}{\sqrt{rr-yy}}$ $= r\sqrt{\frac{r+y}{r-y}}$ = Cotangent of half the Complement of the Latitude AB, whence $z = 2.3025r \times$ Log. Cotangent of half the Complement of the Latitude. And by Correction $z = 2.302585r \times$ *Log. Cot. of half the Co. Lat.* — $2.30258r \times$ *Log. Rad.* and this is the Arch of the Nautical Meridian. But ſince the Meridional Parts in the Tables are expreſſed in Minutes, therefore $z = \frac{2.302585\times180\times60}{3.14159}$ $\times\, \overline{\textit{Log. Cot. } \tfrac{1}{2} \textit{ Co. Lat.} - \textit{Log. Rad.}} = 7915{,}705 \times$ Log. $\frac{\text{Cot. }\frac{1}{2}\text{ Co. Lat.}}{\text{Rad.}} = 7915{,}705\times\text{Log.}\frac{\text{Radius}}{\text{Tan.}\frac{1}{2}\text{Co.Lat.}}$. FIG. 8.

Cor. 1. Hence the Meridional Parts for the Difference of Latitude of two Places is $= 7915{,}7 \times$ *Difference of the Log. Tangents of half their Complements of Latitude.*

Cor. 2. Since Radius : Meridional Difference of Latitude : : ſo Tangent of the Courſe : to the Difference of Longitude; and $\frac{\text{Rad.} = 1}{7915{,}705} =$,000126331, therefore

As ,000126331 :
To *Tangent of the Courſe* : :
So *the Diff. of the Log. Tangents of half the Comp. of the Lat.* :
To *the Difference of Longitude.*

Otherwiſe thus.

Let t = Tang. of the Latitude AB, $DC = x$; then by the Nature of the Circle $yy = rr - xx$ and

FIG. 8. $y\dot{y} = -x\dot{x}$, and $\dot{y} = \frac{-x\dot{x}}{y}$. But by Trigonometry, $\dot{y}\left(\frac{-x\dot{x}}{y}\right) : -\dot{x} :: \text{Rad. } (1) : t = \frac{y}{x}$, and $y = tx$. Likewiſe $x : r :: \dot{y} : \dot{v} = \frac{r\dot{y}}{x}$. And by the Nature of *Mercator*'s Projection, $x : r :: \dot{v}\left(\frac{r\dot{y}}{x}\right) : \dot{z} = \frac{rr\dot{y}}{xx} = \frac{-rr\dot{x}}{xy}$. But $t^2x^2 = yy = rr - xx$, whence $x = \frac{r}{\sqrt{1+tt}}$, and $y = \frac{rt}{\sqrt{1+tt}}$, and $\dot{x} = \frac{-rt\dot{t}}{\overline{1+tt}|^{\frac{3}{2}}}$, and $xy = \frac{rrt}{1+tt}$; therefore $\dot{z} = \frac{-rr \times -rt\dot{t}}{\overline{1+tt}|^{\frac{3}{2}}} \times \frac{1+tt}{rrt} = \frac{r\dot{t}}{\sqrt{1+tt}}$. Therefore, by Form the 9th, $z = 2.30258r \times \text{Log. } t + \sqrt{1+tt}$: But if t is the Tangent, $\sqrt{1+tt}$ is the Secant; and by the Elements of Trigonometry, Secant + Tangent = Cotan. of half the Complement of AB. Therefore $z = 2.30258r \times$ Log. Cotangent of half the Complement of the Latitude AB.

To find the Meridional Parts for the oblate Spheroid.

Let the Semitranſverſe $CA = r$, Semiconjugate $CG = c$, Arch $AB = v$, Ordinate $DB = y$, $CD = x$, Meridional Parts of AB, put $= Z$; and $t =$ Tan. of the Lat. at B, as before.

Then by the Nature of the Figure $ccrr - ccxx = rryy$, and $-ccx\dot{x} = rry\dot{y}$. But by Trigonometry, $\dot{y} : -\dot{x}\left(\frac{rry\dot{y}}{ccx}\right) :: \text{Rad. } (1) : t = \frac{rry}{ccx}$. And, by *Mercator*'s Projection, $x : r :: \dot{v} : \dot{z} = \frac{r\dot{v}}{x}$. FIG. 8.

And ſince $ccxt = rry$, we ſhall have $x = \frac{rr}{\sqrt{rr+cctt}}$, $y = \frac{cct}{\sqrt{rr+cctt}}$, and $\dot{y} = \frac{rrcc\dot{t}}{\sqrt{\overline{rr+cctt}}^{\frac{3}{2}}}$, $\dot{x} = \frac{-rrcct\dot{t}}{\overline{rr+cctt}^{\frac{3}{2}}}$; but $\dot{v} = \sqrt{\dot{x}^2+\dot{y}^2} = \sqrt{\frac{r^4c^4t^2\dot{t}^2 + r^4c^4\dot{t}^2}{\overline{rr+cctt}^3}} = \frac{rrcc\dot{t}\sqrt{1+tt}}{\overline{rr+cctt}\sqrt{rr+cctt}}$; therefore $\dot{z}\left(\frac{r\dot{v}}{x}\right) = \frac{r^3cc\dot{t}\sqrt{1+tt}\times\sqrt{rr+cctt}}{\overline{rr+cctt}\sqrt{rr+cctt}\times rr} = \frac{rcc\dot{t}\sqrt{1+tt}}{rr+cctt}$; or $\dot{z} = \frac{rcc\dot{t}\times\overline{1+tt}}{\overline{rr+cctt}\sqrt{1+tt}} = \frac{r\dot{t}}{\sqrt{1+tt}} + \frac{rcc\dot{t} - r^3\dot{t}}{\overline{rr+cctt}\sqrt{1+tt}}$. Put $rr - cc = ff$, $\sigma =$ Coſec. Latitude, then $\dot{z} = \frac{rt}{\sqrt{1+tt}} - \frac{rff\dot{t}}{\overline{rr+cctt}\sqrt{1+tt}} = \dot{z} - rff \times \frac{t^{-3}\dot{t}}{\overline{cc+rrt^{-2}}\sqrt{1+t^{-2}}} =$ (by Form the 23d) $\dot{z} - rff \times \overline{cc-rr+rrv}^{-1} \times \frac{v^{-\frac{1}{2}}\dot{v}}{-2} = \dot{z} - \frac{rff}{2} \times \frac{v^{-\frac{1}{2}}\dot{v}}{\overline{rr-cc}-rrv}$; whoſe Fluent, by Form the 6th, is $Z = z - \frac{rffL}{2r\sqrt{rr-cc}} \times$ Log:

$r\sqrt{v}$

FIG. 8.

$$\frac{r\sqrt{v}+\sqrt{rr-cc}}{r\sqrt{v}-\sqrt{rr-cc}} = z-\tfrac{1}{2}fL \times \text{Log.}\ \frac{r\sqrt{1+\frac{1}{tt}}+f}{r\sqrt{1+\frac{1}{tt}}-f}$$

$= z-\tfrac{1}{2}fL \times \text{Log.}\ \frac{r\sigma+f}{r\sigma-f}$, becauſe by Trigonometry $\sqrt{1+\frac{1}{tt}}$ or $\frac{\sqrt{1+tt}}{t} = \sigma$. In Minutes $Z = z - 7915.7\frac{f}{r} \times \text{Log.} \sqrt{\frac{\sigma+\frac{f}{r}}{\sigma-\frac{f}{r}}}$. Or putting $\frac{f}{r}$ or $\frac{\sqrt{rr-cc}}{r} = d$, then will $Z = z - \frac{7915.7d}{2} \times \text{Log.}\ \frac{\sigma+d}{\sigma-d}$. Where $\sigma =$ natural Coſecant of the Latitude to the Radius 1, and $z =$ the Meridional Parts for the Sphere.

Or thus.

Since $\frac{1}{\sigma} = y$, therefore $\frac{\sigma+d}{\sigma-d} = \frac{1+dy}{1-dy}$, therefore $Z = z - 7915.7d \times \text{Log.} \sqrt{\frac{1+dy}{1-dy}}$. But by Trigonometry, $\sqrt{\frac{1+dy}{1-dy}} =$ Cotan. $\frac{1}{2}$ Arch whoſe natural Coſine is dy. Therefore, let $y =$ nat. S. Lat. $A =$ Arch whoſe nat. Coſine is dy, to the Rad. 1; and $B =$ Log. Cotan. $\frac{1}{2}A - 10$, as had in the Tables; then $Z = z - 7915.7dB$.

In the ſpheroidal Figure of the Earth, we ſhall have $d = .093$, or thereabouts, according to Sir *I. Newton*; or $d = .148$, according to *Maupertuis.*

Ex.

Ex. 12.

To find the Nature of the Curve which a heavy flexible Line will form itſelf into by its Gravity.

FIG. 9.

Let the Line be ſuſpended on the two fixt Points *D*, *P*, and diſpoſe itſelf into the Curve *DAP*; *A* its Vertex, *AQ* its Axis, and *BC* an Ordinate.

1. The Part of the Curve *ABD* is kept in its Poſition by a certain Force at *A* acting in Direction *AZ* parallel to the Horizon; For if the Line be cut thro' at *A* it will reduce itſelf to a perpendicular Poſition. And this Force acting at *A* is always the ſame whatever Length the Curve be of; For if the Line be cut thro' at *B*, and then the Point *B* faſtned to the Plane; it is evident the Force at *A* is neither greater nor leſs; for the Reſiſtance of the point *B* does the ſame as the Tenſion of the Line in *B* did before; and the Force in *A* or the Tenſion of the Line in *A* muſt remain the ſame.

2. Let $a =$ the given Part of the Line whoſe Weight is equal to the Tenſion of the Line in *A*, and $AC = x$, $BC = y$, $AB = z$, draw the Tangent *BS*, and $BR = BA$, perpendicular to the Horizon, and *RS* parallel to it. The Line *BA* is ſuſtained by three Forces, for its Gravity acts in Direction *BR*, it is drawn at *A* in Direction *AZ* by the Force *a*, and it is ſuſtained in *B* by the Tenſion of the Line in Direction *SB*; and theſe three Forces being as *BR*, *RS*, and *BS*, and $BR = z$ by Conſtruction, therefore the Force $a = RS$: Whence by ſimilar Triangles $BR\,(z) : RS\,(a) :: bo\,(\acute{x}) : Bo\,(\acute{y}) :: \dot{x} : \dot{y}$, and $a\dot{x} = z\dot{y}$, which is one Property of the Curve.

3. Take $Br = Bb$ the Increment of the Curve, draw rn parallel and Bn perpendicular to *BS*, then is $nB = Bo$, and $rn = bo$. Since *BR* is the perpendicular

FIG. dicular Force or Weight of the Line at *B*, *Br* or
9. $\overset{\prime}{z}$ is the Increment thereof, and therefore $\overset{\prime}{x}$ is the Increment of the Tenſion of the Line in *B*, and $\dot{x}$ is the Fluxion of the Tenſion, and therefore the Fluent $x=$ Tenſion or Force acting in Direction nr; But in *A* where $x=0$, this Tenſion $=a$, therefore by Correction, the whole Tenſion drawing in Direction of the Curve is $a+x$; and this is the Force *BS*, as was ſhewn before: Therefore again by ſimilar Triangles $a+x\ (BS):z\ (BR)::\overset{\prime}{z}:\overset{\prime}{x}::\dot{z}:\dot{x}$, whence $a\dot{x}+x\dot{x}=z\dot{z}$, and the Fluent $2ax+xx=zz$, which is another Property of the Curve.

4. If the Point *B* be ſo taken that the Angle *RBS* or *SBC* be half a right Angle, then will *AB* or z be $=a$; for then $\dot{x}=\dot{y}$, and $z=\left(\frac{a\dot{x}}{\dot{y}}=\right)a$.

5. Since $\dot{y}=\frac{a\dot{x}}{z}=\frac{a\dot{x}}{\sqrt{2ax+xx}}=\frac{a\dot{z}}{\sqrt{aa+zz}}=\frac{a\dot{z}}{a+x}$; therefore by Form 9th, $y=2.30258a\times$ Log. $\frac{a+x+\sqrt{2ax+xx}}{a}=2.30258a\times$ Log. $\frac{z+\sqrt{aa+zz}}{a}$; whence the Curve may be eaſily conſtructed.

Ex. 13.

10. *To find the Nature of the Curve BM in which a Body moving (after its Fall thro' AB), it ſhall deſcend equal Spaces in equal Times.*

Let $AB=a$, $BP=x$, $PM=y$, now ſince the Velocities of Bodies are as the Spaces deſcribed in equal Times, and the Squares of the Velocities are as the Heights fallen from; therefore $a:a+x::$ (Square of the Velocity in the Axis at *P*: to Square of the Velocity in the Curve at $M :: Pp^2 : Mm^2 :: \dot{x}^2 : \dot{x}^2 + \dot{y}^2 ::$) $\dot{x}^2 : \dot{x}^2 + \dot{y}^2$, and by Diviſion $a : x :: \dot{x}^2 : \dot{y}^2$; therefore $x\dot{x}^2 = a\dot{y}^2$, or $x^{\frac{1}{2}}\dot{x} = a^{\frac{1}{2}}\dot{y}$, and finding the Fluent $\frac{2}{3}x^{\frac{3}{2}} = a^{\frac{1}{2}}y$, or $ay^2 = \frac{4}{9}x^3$: Therefore the Curve is a ſemi cubical Parabola convex towards BP. FIG. 10.

Ex. 14.

If a Body be projected from any Point A parallel to the horizontal Plane BC, and be urged towards that Plane with a Force which is as any Power of its Height above the Plane; to find the Nature of the Curve it will deſcribe. 11.

Let $AB = r$, $AD = x$, $DP = y$, $v =$ Velocicy acquired in falling through AD, $t =$ Time of falling, $f =$ Force in D or P, which ſuppoſe to be as BD^n.

Now from the Laws of Motion $\dot{v} \propto f\dot{t}$, but $\dot{t} \propto \frac{\dot{x}}{v}$ and $f \propto \overline{r-x}^{n}$, therefore $\dot{v} \propto \frac{\overline{r-x}^{n}\dot{x}}{v}$, and $\dot{v} \propto \frac{\overline{r-x}^{n}\dot{x}}{v}$ or $v\dot{v} \propto \overline{r-x}^{n}\dot{x}$; and finding the Fluent $\frac{vv}{2} \propto \frac{-\overline{r-x}^{n+1}}{n+1}$, but in A, $x = 0$, therefore the Fluent corrected is $vv \propto \frac{r^{n+1} - \overline{r-x}^{n-1}}{n+1}$.

But the Fluxion of the Axis is as the Velocity of a deſcending Body, and the Fluxion of the Ordinate is as the Fluxion of the Time, or as a given Quantity b; therefore $\dot{x} : \dot{y} :: (v =) \sqrt{\frac{r^{n+1} - \overline{r-x}^{n+1}}{n+1}} : b$, FIG. : b, whence $\dot{y} = \dfrac{b\dot{x}}{\sqrt{\dfrac{r^{n+1}}{n+1} - \dfrac{\overline{r-x}^{\,n+1}}{n+1}}}$, or $\dot{y} = \dfrac{b\dot{x}}{\sqrt{r^{n+1} - \overline{r-x}^{\,n+1}}}$, and the Fluent will give the Nature of the Curve.

12. CASE 1. Let $n=1$, then $\dot{y} = \dfrac{b\dot{x}}{\sqrt{2ax-xx}}$; deſcribe the Quadrant AEF; then by the Nature of the Circle, Arch $AE =$ Fluent of $\dfrac{r\dot{x}}{\sqrt{2rx-xx}}$, therefore take y or $DP = \dfrac{b}{r} \times$ Arch AE, and P will be in the Curve required.

CASE 2. Let $n=0$, then $\dot{y} = \dfrac{b\dot{x}}{\sqrt{x}}$, and $y = 2b\sqrt{x}$, or $4bbx = y$; therefore the Curve will be a Parabola, as is well known.

13. CASE 3. Suppoſe $n = -1$. Here we muſt have Recourſe to the original Proceſs itſelf, and there we ſhall have $vv \propto \dfrac{\dot{x}}{r-x}$; deſcribe the Hyperbola HE to the Aſſymptotes AB, BC; then the Area $ADEH =$ Fluent of $\dfrac{\dot{x}}{r-x}$; therefore $\dot{v} \propto \sqrt{ADEH}$, and $\dot{y} = \dfrac{b\dot{x}}{\sqrt{\text{Area } ADEH}}$ for the Nature of the Curve AP.

14. CASE 4. Let $n = -2$, then $\dot{y} = b\dot{x}\sqrt{\dfrac{rr-rx}{x}}$. Let AE be a Cycloid, $AG = v$, $DE = u$, AGB being the generating Circle; then $u = v + \sqrt{rx-xx}$,

and $\dot{u} = \dot{v} + \frac{r - 2x}{2\sqrt{rx - xx}}\dot{x} =$ (becauſe $\dot{v} =$ FIG. 14.
$\frac{r\dot{x}}{2\sqrt{rx - xx}}$,) $\dot{x}\sqrt{\frac{r - x}{x}}$. But $\dot{y} = br^{\frac{1}{2}} \times \dot{x}\sqrt{\frac{r - x}{x}}$, whence $y = bu\sqrt{r}$, therefore take $DP = b\sqrt{r} \times DE$, and *P* will be in the Curve.

CASE 5. Let $n = -3$; then $\dot{y} = \frac{br \times \overline{r\dot{x} - x\dot{x}}}{\sqrt{2rx - xx}}$, whence $y = br\sqrt{2rx - xx}$, and the Curve is an Ellipſis whoſe Semi-axis is *AB*.

Ex. 15.

To know whether a given Curve Line be concave or convex towards its Axis, in any given Point thereof. 15.

Let *AB* be the Axis, *BC* any Ordinate, take *DB* $=$ *BF*, and draw the Ordinates *De*, *Fo* infinitely near *BC*; draw *en* and *Cg* parallel to *AB*, and produce *eC* to *k*. Now by the Nature of Curvature, if the Curve be concave towards the Axis, in the Point *C*, and the Ordinates increaſe, then the Point *o* of the Curve will fall between *k* and *g*, and therefore the Increment *go* is leſs than *nC*, whence *ok* or *go*—*gk* that is *go*—*nC*, will be negative; but *ok* is the ſecond Moment of the Ordinate, and that is as the ſecond Fluxion when *Fk* and *De* approach to, and coincide with *BC*; therefore if the Curve be concave towards the Axis the ſecond Fluxion of the Ordinate will be negative. And the contrary will happen if the Curve 16.
is convex towards the Axis. Wherefore

Let the Axis $AB = x$, Ordinate $BC = y$, then compute the Value of $\ddot{y}$ by the Nature of the Curve, and ſubſtitute Numbers for all the Quantities if there be Occaſion, then if its Value comes out negative it is concave in that Point, if affirmative it is convex towards the Axis.

Ex. 16.

FIG. 17. *A Line being inflated into the Form of a Curve, and kept in that Form by a Force or Preſſure acting perpendicularly upon every Point of the Curve; to find the Proportion between the Tenſion of the Curve, and the Force acting perpendicularly upon it.*

Let *AB*, *BC* be two equal Particles of the Curve, and let the Force acting on the Particle *B* reduce it to the Poſition *ABC*; compleat the Parallelogram *ABCE*, and draw *AD*, *CD*, perpendicular to *AB*, *CB*; and then the four Points, *A*, *B*, *C*, *D*, will lie in a Circle. Let *AB* or $BC = \acute{z}$, *AD* or $CD = r$: Now by Mechanics the Point *B* is acted upon by three Forces *BA*, *BC*, *BE*; *BA* or *BC* is the Tenſion of the Curve, and *BE* is the Force acting perpendicularly againſt the Curve, therefore theſe are to one another as *BA* to *BE*, or (by ſimilar Triangles) as $\frac{1}{2}$ *BD* to $\acute{z}$. Let now the Points *A*, *C*, approach to *B* and coincide with it, and $\frac{1}{2}$ *BD* becomes $\frac{1}{2}r$, the Radius of Curvature; and $\acute{z}$ is the Particle of the Curve, whereon the Preſſure acts: And therefore the Force acting perpendicularly on any Particle of the Curve is to the Tenſion of the Curve, as that Particle of the Curve is to the Radius of Curvature in that Point.

COR. Therefore if the Particle of the Curve and its Tenſion be given, the Force acting againſt that Particle is reciprocally as the Radius of Curvature; that is, directly as the Curvature.

Ex. 17.

EAF is a Curve Line ſupporting a Fluid; to find the Nature of the Curve. FIG. 18.

Let the Axis of the Figure $CA=b$, $CD=x$, $DB=y$, $AB=z$, let $\dot z$ be given: By reaſon of the Fluidity of the Water, the Tenſion of the Curve is equal in all Points, and therefore by the forgoing Example, the Preſſure at *B* is reciprocally as the Radius of Curvature in *B*. But by the Laws of Hydroſtatics the Preſſure at *B* is as the Height *DB* or *y*, and (by Prob. V. Sect II.) the Radius of Curvature is $\frac{\dot z\dot y}{\ddot x}$, therefore $\frac{1}{y} \propto \frac{\dot z\dot y}{\ddot x}$, and aſſuming the given Quantity $\frac{aa}{2}$, then $\frac{aa}{2y} = \frac{\dot z\dot y}{\ddot x}$, or $\frac{aa\ddot x}{2} = y\dot y\dot z$, and the Fluent $aa\dot x = y^2\dot z$; but in *A*, $\dot x = \dot z$ and $y=b$; therefore the Fluent corrected is $aa\dot x - aa\dot z = \overline{yy-bb}\times\dot z$, or $aa\dot x = \overline{aa+yy-bb}\times\sqrt{\dot x^2+\dot y^2}$, which reduced gives $\dot x = \frac{\overline{aa-bb+yy}\times\dot y}{\sqrt{\begin{smallmatrix}2aabb\\-b^4\end{smallmatrix}\begin{smallmatrix}+2bb\\-2aa\end{smallmatrix}y^2-y^4}}$ for the Equation of the Curve ſought.

Cor. Draw *EK* perpendicular, and *IK* parallel to *ED*, and alſo the Tangent *EI*; than if $EK=p$, $EI=q$, $s=$ Area *EAC*, which is as its Weight, then (by Mechanics) $p:q::s:\frac{qs}{p}=$ Tenſion of the Curve, whence (by the foregoing Example) ſince $y\dot z=$ Preſſure at *B*, it will be $y\dot z : \frac{qs}{p} : \dot z : \frac{\dot z\dot y}{\ddot x}$, and thence $\frac{qs\ddot x}{p} = y\dot y\dot z = \frac{aa}{2}\ddot x$, and therefore $\frac{2qs}{p} = aa$.

Ex. 18.

FIG. *To find an infinite Series, or ſeveral ſuch, whoſe Sum may be had.*

Aſſume $\dot{v} = \frac{\dot{x}}{1-x}$ or $\frac{\dot{x}}{1-x^2}$ or $\frac{\dot{x}}{1-x^3}$ &c. For Example, let $\dot{v} = \frac{\dot{x}}{1-x} = 1 + x + x^2 + x^3$ &c. $\times : \dot{x}$, therefore $v = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4$ &c.

Suppoſe the Fl: $vx^n\dot{x} = \frac{vx^{n+1}}{n+1} + q$; then by putting this into Fluxions, we ſhall find $\dot{q} = \frac{-x^{n+1}}{n+1}\dot{v} = \frac{-x^{n+1}}{n+1} \times \frac{\dot{x}}{1-x} = \frac{x^{n+1}\dot{x}}{\overline{n+1} \times \overline{x-1}} = \frac{1}{n+1} \times : x^n\dot{x} + x^{n-1}\dot{x} + x^{n-2}\dot{x} + x^{n-3}\dot{x}$, &c. to $\frac{\dot{x}}{x-1}$ or $-\dot{v}$; therefore $q = \frac{1}{n+1} \times : \frac{x^{n+1}}{n+1} + \frac{x^n}{n} + \frac{x^{n-1}}{n-1}$ &c. $+ x - v$. Whence Fl: $vx^n\dot{x} = \frac{vx^{n+1}}{n+1} + \frac{1}{n+1} \times : \frac{x^{n+1}}{n+1} + \frac{x^n}{n} + \frac{x^{n-1}}{n-1}$, &c. $+ x - v$.

But Fl: $vx^n\dot{x}$ is alſo $=$ F: $x^n\dot{x} \times : x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}$ &c. $= \frac{x^{n+2}}{n+2} + \frac{x^{n+3}}{2 \times \overline{n+3}} + \frac{x^{n+4}}{3 \times \overline{n+4}}$, &c. *ad infinitum.*

Therefore $\frac{x^{n+2}}{n+2} + \frac{x^{n+3}}{2 \times \overline{n+3}}$, &c. $= \frac{vx^{n+1} - v}{n+1} + \frac{1}{n+1} \times : \frac{x^{n+1}}{n+1} + \frac{x^n}{n}$, &c. to x.

Put $\frac{vx^{n+1} - v}{n+1} = 0$, and then $x = 1$.

Therefore the foregoing Series will become $\frac{1}{n+2} + \frac{1}{2\times\overline{n+3}} + \frac{1}{3\times\overline{n+4}} + \frac{1}{4\times\overline{n+5}}$ &c. *ad infinitum* $= \frac{1}{n+1} \times : \frac{1}{n+1} + \frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2}$ &c. to 1.

Or, which is the ſame Thing, $\frac{1}{n+1} \times : 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ &c. to $\frac{1}{n+1}$ = Sum of the infinite Series $\frac{1}{n+2} + \frac{1}{2\times\overline{n+3}} + \frac{1}{3\times\overline{n+4}} + \frac{1}{4\times\overline{n+5}}$ &c. *ad infinitum.*

Now if n be put ſucceſſively equal to any Integer Numbers, there will be had ſo many infinite Series, each of which will be equal to the Sum of a given finite Series.

For Example, ſuppoſe $n = 1$, then $\frac{1}{2} \times \overline{1 + \frac{1}{2}}$, or $\frac{3}{4}$ = to the infinite Series $\frac{1}{1.3} + \frac{1}{2.4} + \frac{1}{3.5} + \frac{1}{4.6}$, &c.

Moreover, whatever be the Value of x, if v or the F: $\frac{\dot{x}}{1-x}$ be known, we ſhall always have the finite Series $\frac{1}{n+1} \times : vx^{n+1} - v + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4$ . . . to $\frac{1}{n+1}x^{n+1}$ = to the Series $\frac{x^{n+2}}{1\times\overline{n+2}} + \frac{x^{n+3}}{2\times\overline{n+3}} + \frac{x^{n+4}}{3\times\overline{n+4}} + \frac{x^{n+5}}{4\times\overline{n+5}}$, &c. *ad infinitum.*

The like may be done by aſſuming any other Quantity for v; as for Example, let $v = \text{F}: \frac{\dot{x}}{1-x^3}$, or $\dot{v} = \frac{\dot{x}}{1-x^3} = \dot{x} + x^3\dot{x} + x^6\dot{x} + x^9\dot{x}$, &c. and $v = x + \frac{1}{4}x^4 + \frac{1}{7}x^7 + \frac{1}{10}x^{10}$, &c.

Suppoſe, as before, $\text{F}: vx^n\dot{x} = \frac{vx^{n+1}}{n+1} + q$; then $\dot{q} = \frac{x^{n+1}}{n+1} \times \frac{\dot{x}}{x^3-1} = \frac{1}{n+1} \times x^{n-2}\dot{x} + x^{n-5}\dot{x} \ldots + \dot{x} - \dot{v}$, and $q = \frac{1}{n+1} \times : \frac{x^{n-1}}{n-1} + \frac{x^{n-4}}{n-4} + \frac{x^{n-7}}{n-7} \ldots + x - v = \frac{1}{n+1} \times : -v + x + \frac{x^4}{4} + \frac{x^7}{7} + \frac{x^{10}}{10} \ldots + \frac{x^{n-1}}{n-1}$.

Again, $\text{F}: vx^n\dot{x} = \text{F}: x^n\dot{x} \times : x + \frac{1}{4}x^4 + \frac{1}{7}x^7$ &c. $= \frac{x^{n+2}}{n+2} + \frac{x^{n+5}}{4 \times \overline{n+5}} + \frac{x^{n+8}}{7 \times \overline{n+8}}$, &c. therefore the finite Series $\frac{1}{n+1} \times : vx^{n+1} - v + x + \frac{x^4}{4} + \frac{x^7}{7} \ldots + \frac{x^{n-1}}{n-1}$ = the Sum of the Series $\frac{x^{n+2}}{n+2} + \frac{x^{n+5}}{4 \times \overline{n+5}} + \frac{x^{n+8}}{7 \times \overline{n+8}}$, &c. *ad infinitum*; where n is any Number in this Series 2, 5, 8, 11, &c. and in general,

If $\dot{v} = \frac{\dot{x}}{1-x^m}$, then n muſt be ſuch that $n+1$ may be diviſible by m. And if you make $vx^{n+1} - v = 0$, and $x = 1$, as before, you may get as many particular Series as you pleaſe.

Ex. 19.

To Expreſs the Fluent of a ſimple Expreſſion $ax^b\dot{x}$, by an infinite Series; and from thence to ſum up that Series. FIG. 18.

Put $ax^b\dot{x} = \dot{F}$ and expreſs the Index by two Parts as $ax^{b+c}x^{-c+1}\dot{x}$, then aſſume $F = \frac{ax^{b+c}}{b+c}x^{-c+1} + s$, then $\dot{F}$ or $ax^b\dot{x} = ax^b\dot{x}\,\overline{-c+1} \times \frac{a}{b+c}x^b\dot{x} + \dot{s}$, whence $\dot{s} = \overline{c-1} \times \frac{a}{b+c}x^b\dot{x} = \overline{c-1} \times Ax^b\dot{x}$. Again,

Aſſume $s = \frac{\overline{c-1} \times A}{b+d}x^{b+d}x^{-d+1} + t = Bx^{b+d}x^{-d+1} + t$; whence $\dot{t} = \overline{d-1} \times Bx^b\dot{x}$.

Aſſume $t = \frac{\overline{d-1} \times B}{b+e}x^{b+e}\,x^{-e+1} + v = Cx^{b+e}x^{-e+1} + v$; then $\dot{v} = \overline{e-1} \times Cx^b\dot{x}$.

Aſſume $v = \frac{e-1}{b+f}Cx^{b+f}x^{-f+1} + w$, &c.

Whence F or Fl: $ax^b\dot{x} = x^{b+1} \times : \frac{a}{b+c} + \frac{c-1}{b+d}A + \frac{d-1}{b+e}B + \frac{e-1}{b+f}C + \frac{f-1}{b+g}D +$ &c. $= \frac{ax^{b+1}}{b+1}$; where A, B, C, &c. are the preceding Terms. Therefore taking c, d, e, &c. any Quantities at Pleaſure, the Series will be known, and the Sum of it given.

1. Suppoſe $a=1$, b, c, d, e, &c. $= 2$. Then Fl: $x^2\dot{x} = x^3 \times : \frac{1}{4} + \frac{1}{4}A + \frac{1}{4}B + \frac{1}{4}C$, &c. $= \frac{x^3}{3}$. Therefore $\frac{1}{4} + \frac{1}{4}A + \frac{1}{4}B$, &c. $= \frac{1}{3}$.

FIG. 2. Suppoſe b, c, d, e, &c. $= m$. Then Fl: $ax^{m}\dot{x} = x^{m+1} \times : \frac{a}{2m} + \frac{m-1}{2m}A + \frac{m-1}{2m}B + \frac{m-1}{2m}C$, &c. $= \frac{ax^{m+1}}{m+1}$; and $\frac{1}{m+1} = \frac{1}{2m} + \frac{m-1}{2m}A + \frac{m-1}{2m}B$, &c.

3. Let $a=1$, $b=n-1$, $c=2$, $d=3$, $e=4$, $f=5$, &c. Then F: $x^{n-1}\dot{x} = x^{n} \times : \frac{1}{n+1} + \frac{1}{n+2}A + \frac{1}{n+3}B + \frac{1}{n+4}C$, &c. $= \frac{x^{n}}{n}$. Whence alſo $\frac{1}{n+1} + \frac{1}{n+2} \times A + \frac{1}{n+3}B + \frac{1}{n+4}C$, &c. $= \frac{1}{n}$.

Ex. 20.

213. *To find the Curve which a flexible Line QAR is put into by the Wind, or any Fluid moving againſt it.*

Let AE be the Axis, and BE, Cf Ordinates infinitely near, Bd parallel to AE. Call AE, x; EB, y; AB, z; Bd, $\overset{\prime}{x}$; Cd, $\overset{\prime}{y}$; BC, $\overset{\prime}{z}$; and let $\overset{\prime}{z}$ be given. The Force of the Fluid acting perpendicularly againſt the Particle of the Curve $\overset{\prime}{z}$, is as the Quantity of the Fluid acting on it, and the Sine of Incidence, that is as $\overset{\prime}{y} \times \frac{\overset{\prime}{y}}{\overset{\prime}{z}}$, or $\frac{\overset{\prime}{y}^{2}}{\overset{\prime}{z}}$, that is as $\frac{\dot{y}^{2}}{\dot{z}}$; And (by Ex. 16. before) that Force will be as the Curvature in B, or reciprocally as the Radius of Curvature, that is (by Prob. V. Sect. II.) as $\frac{\dot{z}\dot{x}}{-\ddot{y}}$; there-

fore $\frac{\dot{y}^2}{\dot{z}} \propto \frac{-\ddot{y}}{\dot{z}\dot{x}}$, or $\frac{\dot{y}^2}{\dot{z}} = \frac{-a\ddot{y}}{\dot{x}}$; therefore $\frac{\dot{x}}{\dot{z}} =$ FIG. 213.
$\frac{-a\ddot{y}}{\dot{y}^2}$, and the Fluent is $\frac{x}{\dot{z}} = \frac{a}{\dot{y}}$; but in A, $\dot{y} = \dot{z}$, and $\dot{x} = 0$, therefore the corrected Fluent is $\frac{x}{\dot{z}} = \frac{a}{\dot{y}} - \frac{a}{\dot{z}}$, whence $x\dot{y} = a\dot{z} - a\dot{y}$, therefore $\overline{a+x} \times \dot{y} = a\dot{z}$, and $\overline{a+x}^2 \times \overline{\dot{z}^2 - \dot{x}^2} = a^2\dot{z}^2$; or $\overline{2ax + xx} \times \dot{z}^2 = \overline{a+x}^2 \times \dot{x}^2$, and $\dot{z} = \frac{\overline{a+x} \times \dot{x}}{\sqrt{2ax+xx}}$; and the Fluent $z = \sqrt{2ax + xx}$, an Equation to the Catenary.

Ex. 21.

To find the Curve of Traction to a right Line. That 228.
is, if a heavy Body be placed at A, and a Cord AB of a given Length fastened to it, and the End B drawn along the right Line BD; to find the Curve described by the Body A.

Let AB be perpendicular to BD, and let AE be the Curve described by A. Draw the Ordinates EC, ec infinitely near each other, and ⊥ to BD, and draw the Tangents ED, ed; and Do ⊥ to ed. Then put AB, ED or $ed = b$, $AE = z$, $BC = x$, $EC = y$, $CD = s$; then $bb = ss + yy$, and $2s\dot{s} = -2y\dot{y}$. And by similar Triangles, $y : b :: -\dot{y} : \dot{z}$, and $y : \sqrt{bb - yy} :: -\dot{y} : \dot{x}$, whence

$$\dot{z} = \frac{-b\dot{y}}{y} = \frac{bs\dot{s}}{bb - ss}, \text{ and } \dot{x} = \frac{-\dot{y}}{y}\sqrt{bb-yy} = \frac{s^2\dot{s}}{bb-ss}.$$

And by Form 2d and 4th, $z = -bL \times \text{Log:}\, y = -\frac{bL}{2} \times \text{Log:}\, \overline{bb - ss}$. But in A, $y = b$, or $s = 0$, therefore the correct Fluent is $z = 2.30258b \times \text{Log:}\, \frac{b}{y} = \frac{2.30258b}{2} \times \text{Log:}\, \frac{bb}{bb - ss}$.

Bu

FIG. 228. But $\frac{-\dot{y}}{y}\sqrt{bb-yy} = \frac{-\dot{y}\times\overline{bb-yy}}{y\sqrt{bb-yy}} = bb \times \frac{-y^{-2}\dot{y}}{\sqrt{-1+bby^{-2}}} + \frac{y\dot{y}}{\sqrt{bb-yy}}$; and by Form the 9th and 3d, the Fluent is $= bL \times \text{Log:}\, \frac{b}{y} + \sqrt{-1+\frac{bb}{yy}} : - \sqrt{b^2-y^2}$.

Alſo $\frac{s^2\dot{s}}{bb-ss} = -\dot{s} + \frac{bb\dot{s}}{bb-ss}$; and by Form 6th, the Fluent is $-s + \frac{bL}{2} \times \text{Log:}\frac{b+s}{b-s}$. Therefore $x = -\sqrt{bb-yy} + 2.30258b \times \text{Log:}\, \frac{b+\sqrt{bb-yy}}{y} : = \frac{2.30258b}{2} \times \text{Log:}\, \frac{b+s}{b-s} : -s$.

Ex. 22.

229. *To find the Curve of Traction to a Circle; or the Curve EN deſcribed by a Body N, drawn along by a Cord DN of a given Length, while the End D paſſes through the Periphery of the Circle BAD.*

Through *N*, *n*, two Points in the Curve infinitely near, draw *CA*, *Ca*, and the two Tangents *ND*, *nd*; draw *DF* ⊥ to *CA*, and *Do* ⊥ to *nd*; and draw *CD*, and alſo *CEB*.

Then let $CB = r$, EB or $ND = b$, $BA = x$, $CN = y$, $EN = z$, $AD = v$, $DF = s$, $NF = P$. Then in the Triangle *CDN*, $rr = bb + yy + 2Py$, whence $P = \frac{rr-bb-yy}{2y} = \sqrt{bb-ss}$. Then ſince $ND = nd$, Nn or $\dot{z} = do$; alſo $\dot{v} = ad - AD = Dd - Aa = Dd - \dot{x}$, and $Dd = \dot{x} + \dot{v}$, and by the Nature of the Circle $\dot{v} = \frac{r\dot{s}}{\sqrt{rr-ss}}$. By ſimilar Sectors, $r : \dot{x} :: y$

$:: y : nm = \frac{y\dot{x}}{r}$. And the Triangles DFN, nmN being ſimilar, $s : b :: \frac{y\dot{x}}{r} : \dot{z} = \frac{by\dot{x}}{rs}$; and by Trigogometry $b : s\ (S.DCN) :: y : \frac{sy}{b} = S.CDN$ or dDo; and $r : \dot{x}+\dot{v}\ (Dd) :: \frac{sy}{b} : (do$ or$)\ \dot{z} = \frac{sy\dot{x}+sy\dot{v}}{rb} = \frac{by\dot{x}}{rs}$, and $\frac{s\dot{x}+s\dot{v}}{b} = \frac{b\dot{x}}{s}$, and $ss\dot{x} + ss\dot{v} = bb\dot{x}$; therefore $\dot{x} = \frac{ss\dot{v}}{bb-ss} =$

FIG. 229.

$$\frac{rs^2\dot{s}}{\overline{bb-ss}\sqrt{rr-ss}} = \frac{-r\dot{s}}{\sqrt{rr-ss}} + \frac{rbb\dot{s}}{\overline{bb-ss}\sqrt{rr-ss}}.$$

By Form 10th, the Fluent of $\frac{r\dot{s}}{\sqrt{rr-ss}} =$ Arch whoſe Radius is r and Sine s, and is therefore $=$ Arch AD; and $\frac{rbb\dot{s}}{\overline{bb-ss}\sqrt{rr-ss}} =$

$$\frac{rbbs^{-3}\dot{s}}{\overline{-1+bbs^{-2}}\sqrt{-1+rrs^{-2}}} = \text{(by Form the 23d)}$$

$$\frac{rb^2v^{-\frac{1}{2}}\dot{v}}{f\times\overline{p+\frac{bv}{f}}} = \frac{rbb}{-2rr} \times \frac{v^{-\frac{1}{2}}\dot{v}}{p+\frac{bb}{rr}v} =$$

$$\frac{rbb}{2} \times \frac{v^{-\frac{1}{2}}\dot{v}}{-prr-bbv} = \frac{rbb}{2} \times \frac{v^{-\frac{1}{2}}\dot{v}}{\overline{rr-bb}-bbv};$$

whoſe Fluent (by Form the 6th) $= \frac{bbr}{2} \times$

$$\frac{L}{b\sqrt{rr-bb}} \times \text{Log:}\ \frac{\sqrt{rr-bb}+b\sqrt{v}}{\sqrt{rr-bb}-b\sqrt{v}} =$$

FIG. 229.

$$\frac{brL}{2\sqrt{rr-bb}} \times \text{Log:}\ \frac{\sqrt{rr-bb}+b\sqrt{\frac{rr-ss}{ss}}}{\sqrt{rr-bb}-b\sqrt{\frac{rr-ss}{ss}}} :$$

$$= \frac{brL}{2\sqrt{rr-bb}} \times \text{Log:}\ \frac{s\sqrt{rr-bb}+b\sqrt{rr-ss}}{s\sqrt{rr-bb}-b\sqrt{rr-ss}}.$$

Therefore $x = \frac{brL}{2\sqrt{rr-bb}} \times$ Log:

$$\frac{s\sqrt{rr-bb}+b\sqrt{rr-ss}}{s\sqrt{rr-bb}-b\sqrt{rr-ss}} - \text{Arch } AD.$$

Again $P : b :: \dot{y} : \dot{z} = \frac{b\dot{y}}{P} = \frac{2by\dot{y}}{rr-bb-yy}$; and $z = 2.30258b \times -$ Log: $rr-bb-yy$: and corrected $z = 2.3025b \times \text{Log:}\ \frac{2rb-2bb}{rr-bb-yy}$.

SCHOLIUM.

Since $\dot{y} = \frac{rr-bb-yy}{2by}\dot{z}$, therefore when y is a Maximum, $\dot{y}$ or $\frac{rr-bb-yy}{2by}\dot{z} = 0$, and $rr-bb-yy = 0$; whence $y = \sqrt{rr-bb}$; then $P=0$, and $s=b$. Therefore the Curve can approach the Circle no nearer than $r-\sqrt{rr-bb}$; therefore if a Circle be deſcribed from the Center C, with the Radius $\sqrt{rr-bb}$, this Circle will be an Aſſymptote to the Curve.

And let theſe few Examples here ſuffice, to ſhew the Application of the Rules, to the Calculation of particular Problems. We will now proceed to the Reſolution of more general Problems in Mathematics and Natural Philoſophy.

SECT.

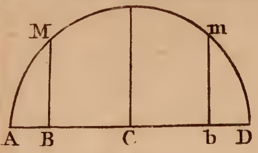
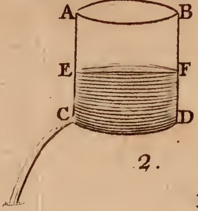


Fig.1.



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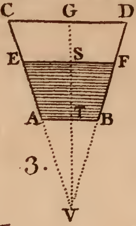

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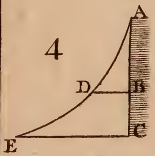

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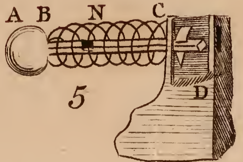

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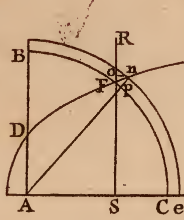

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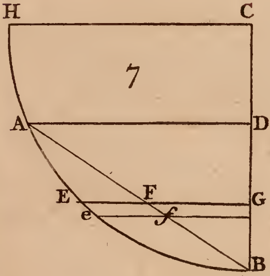

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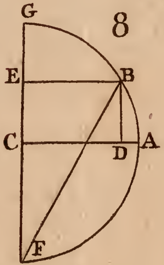

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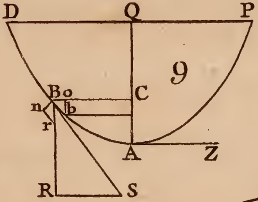

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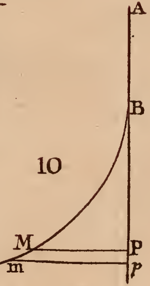

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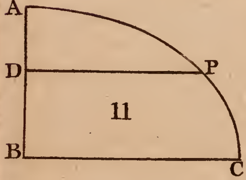

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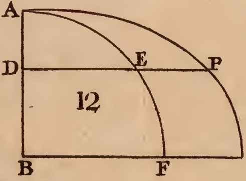

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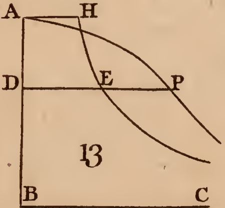

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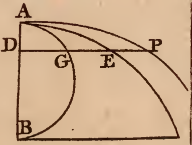

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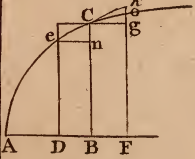

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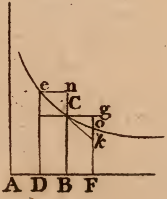

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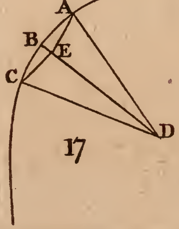

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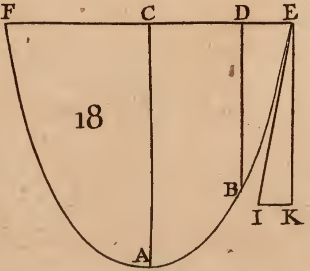


18

I.Baſire ſculp. Fluxions

Pl I. p.166.

SECT. II.

The Inveſtigation and Solution of ſome of the moſt general and uſeful Problems in the Mathematics.

PROB. I.

To determine the Maxima and Minima of Quantities.

WHEN a Quantity is required to be the greateſt or leaſt poſſible, under certain Conditions, it is called a Maximum or Minimum; and at that Moment it neither increaſes nor decreaſes, and therefore its Fluxion is nothing. Now ſince any Quantity is a Maximum or Minimum, when its Fluxion is nothing, upon the Suppoſition of only one variable Quantity therein, and the ſame is true when any other Quantity alone is ſuppoſed variable: Conſequently when there are ſeveral variable Quantities in the Maximum or Minimum, then each of theſe Fluxions muſt be ſeparately equal to nothing. Therefore

1. *Put the conſtant Quantity m for the Maximum or Minimum required; and get an Equation involving m, by*

by Help of which and other given Equations, exterminate as many of the variable Quantities as you pleaſe, if there be ſeveral: Then put the remaining Equations into Fluxions, and for each Equation exterminate one Fluxion, till you have but one Equation; in which (brought all to one Side) make the Sum of all the Terms multiplying each particular Fluxion, ſeparately $=0$, *and you will get as many Equations, which, together with thoſe at firſt given, will determine all the unknown Quantities.*

Sometimes we muſt make the Logarithm of the Quantity a Maximum or Minimum, or the Log: of the Logarithm of it, &c. *and put its Fluxion* $= 0$. *This Proceſs is ſometimes neceſſary in exponential Quantities.* For if the Quantity itſelf be a *Max.* or *Min.* its Log. will be ſo too.

2. *But when the Fluxion of any ſingle Quantity is found* $=0$, *then that Quantity itſelf is either a Maximum or Minimum, or a ſtanding Quantity. Or if an impoſſible Equation come out, the Quantity will have no Maximum or Minimum but what is infinite. Oftentimes the Equation will have ſeveral Roots, all which muſt be ſeparately tried, to ſee which of them will anſwer the Conditions of the Queſtion, and give the Maximum or Minimum required; and this is done by putting the ſingle variable Quantity in the Maximum or Minimum equal to ſeveral ſucceſſive Values expreſſed in Numbers.*

230. When the Nature of a Curve, as *ECD*, is required, which ſhall contain ſome *Maximum or Minimum*; then ſuppoſe two Points of the Curve *F* and *G* to be given; then we muſt find an intermediate Point *C*, ſo that the Part between *F* and *G*, or *FH* and *GI*, may be a *Maximum* or *Mininum*. For if the Point *C* be not ſo ſituated, that the Part between *FH* and *GI* be a *Maximum* or *Minimum*, 'tis plain the whole cannot be ſo. Therefore,

3. *Draw two Ordinates FH, GI infinitely near, intercepting an infinitely ſmall given Part of the Maxi-*

mum or Minimum, and alſo of the given Quantity mentioned in the Problem, to find an intermediate Point C, draw the Ordinate P C, an arithmetic Mean between the other two F H, G I. This ordinate divides theſe infinitely ſmall Quantities into two Parts; then the Fluxion of the Sum of each muſt be put $=0$, *which gives two Equations, from whence the Nature of the Curve will be had.* FIG. 230.

In Caſes more compounded, where the Problem involves ſeveral Conditions, and the Nature of the Curve is required. Let *EFD* be the Curve, and let any Point *F* be given therein, and let *C* be any other Point in the Curve, infinitely near *F*. Draw the Ordinates *FH*, *CP*; and *Fm* parallel to the Axis *AB*. To find the Situation of the Point *F*, ſo that $A\times FC-B\times Fm$ may be $=$ *a Maximum or Minimum*; ſuppoſing *A*, *B*, conſtant Quantities. Let $Fm=s$, $Fc=t$, $Cm=n$, a given Quantity. Then $A\sqrt{nn+ss}-Bs=m$, in Fluxions $\frac{As\dot{s}}{\sqrt{nn+ss}}-B\dot{s}=0$, whence $As=B\sqrt{nn+ss}=Bt$, for the Nature of the Curve *EFD*. Now ſince *HF* (or *y*) is given in reſpect of the Point *F*; therefore *A*, *B* may be made up of *y* and any given Quantities. And the ſame Thing holds good of any other Point of the Curve. Therefore when $At-Bs=m$, the Equation of the Curve is $As=Bt$. But ſince $At-Bs=$ *Max.* or *Min.* therefore the Sum of all the $At-Bs=$ *Max.* *or Min.* that is $F:A\dot{z}-F:B\dot{x}=m$, where *FC* or $t=\dot{z}$, *Fm* or $s=\dot{x}$. And if either $F:A\dot{z}$ or $F:Bx$ be given, the other is either a *Maximum or Minimum.*

Whence the following Rule,

4. *Let* $AH=x$, $HF=y$, $EF=z$, *and let* $\dot{y}$ *be given; and ſuppoſe* $F:A\dot{z}-F:B\dot{x}=m$. *Where A and*

FIG. 230. *and B are given Quantities, or made up of y and given Quantities, or ſuch as denote a Maximum or Minimum. And all the Conditions of the Queſtion being properly expreſſed; collect all the Quantities affected with $\dot{z}$, for the Value of A; and all the Quantities affected with $\dot{x}$, for the Value of B; then theſe Values being ſubſtituted in the Equation $A\dot{x} = B\dot{z}$, inſtead of A and B, gives the Equation of the Curve. And the ſame holds good if you make*

$$F : \frac{\dot{z}}{B} - F : \frac{\dot{x}}{A} = a \text{ Max. or Min.}$$

SCHOL.

Concerning curvilineal Spaces, it is in Effect the ſame Thing to ſeek the greateſt Area contained under a given Perimeter, as to ſeek a given Area under the leaſt Perimeter. The ſame will hold in Reſpect of Solids and their Surfaces.

EXAMPLE I.

To find $xz+yy$ a Minimum, ſo that $x+y+z=b$.

Put $xz+yy=m$, and expunging z, $bx-xx-xy+yy=m$. In Fluxions $b\dot{x}-2x\dot{x}-y\dot{x}-x\dot{y}+2y\dot{y}=0$, and taking the homologous Terms, $b\dot{x}-2x\dot{x}-y\dot{x}=0$, and $2y\dot{y}-x\dot{y}=0$, whence $2x+y=b$, and $x=2y$; and therefore $y=\frac{1}{5}b$, $x=\frac{2}{5}b$, $z=\frac{2}{5}b$.

Or thus;

Since $xz+yy=m$, in Fluxions $x\dot{z}+z\dot{x}+2y\dot{y}=0$, alſo from the Equation $x+y+z=b$, we have $\dot{x}+\dot{y}+\dot{z}=0$; and expunging $\dot{z}$, we get $-x\dot{x}-x\dot{y}+z\dot{x}+2y\dot{y}=0$, and therefore $z\dot{x}-x\dot{x}=0$, or $z=x$; and $2y\dot{y}-x\dot{y}=0$, or $x=2y$, whence x, y, z are found the ſame as before.

Note, In this Example if x be given there will be a Minimum, but the Maximum is infinite; and if y be given there will be only a Maximum: Therefore in general there is no Maximum or Minimum but what is infinite.

Ex. 2.

To find x the greateſt in this Equation $x^3 - ax^2 + axy - y^3 = 0$. FIG.

For x write m, and the Equation is $m^3 - am^2 + amy - y^3 = 0$; In Fluxions $am\dot{y} - 3y^2\dot{y} = 0$, whence $3y^2 = am$, by which and the former Equation, m and y are determined.

Ex. 3.

To find a Cylinder of a given Solidity b, with the leaſt whole Surface.

Let y = Altitude, x = Diameter of the Baſe, $c = 3.1416$, then $\frac{cx^2y}{4} = b$, and $\frac{cxx}{2}$ = Sum of the Baſes, cxy = convex Surface $= \frac{4b}{x}$; therefore $\frac{cx^2}{2} + \frac{4b}{x} = m$: In Fluxions $cx\dot{x} - \frac{4b\dot{x}}{xx} = 0$, which reduced gives $x = \sqrt[3]{\frac{4b}{c}}$, and thence $y = \sqrt[3]{\frac{4b}{c}} = x$.

Ex. 4.

In the Semicircle ABC, to find the Rectangle ADB a Maximum. 19.

Let $AC = a$, $AD = x$, $DB = \sqrt{ax - xx}$, then $x\sqrt{ax - xx} = m$, or $ax^3 - x^4 = m^2$: In Fluxions $3ax^2\dot{x} - 4x^3\dot{x} = 0$, whence $x = \frac{3}{4}a$.

Ex. 5.

Given the Baſe AB and Perpendicular CD, in a Triangle ACB, to find the Angle ACB, the greateſt poſſible. 20.

Biſſect AB in E, and let $CD = p$, $AE = q$, $ED = y$; and by Trigonometry (CB) $\sqrt{pp + qq - 2qy + yy}$:

FIG. 20. (Radius) $R :: p : \frac{pR}{\sqrt{pp+qq-2qy+yy}} = S.<B$; and ($AC$) $\sqrt{pp+\overline{q+y}^2} : (S.<B) \frac{pR}{\sqrt{pp+\overline{q-y}^2}}$ $:: 2q : \frac{2pqR}{\sqrt{pp+\overline{q+y}^2} \times \sqrt{pp+\overline{q-y}^2}} = S.<C$ $= m$, and $p^4 + 2ppqq + q^4 + 2p^2y^2 - 2q^2y^2 + y^4 = \frac{4ppqqR^2}{mm}$: In Fluxions $4ppy\dot{y} - 4qqy\dot{y} + 4y^3\dot{y} = 0$, and $y^3 = \overline{qq-pp} \times y$, and one of the Roots is $y = 0$; the other Quantity $y^2 = qq - pp$ is an impoſſible Equation when p is bigger than q; wherefore the Point D falls in E.

Ex. 6.

21. *The Point P being given in the Tranſverſe of the Ellipſis ADR; to find PB the neareſt Diſtance to the Curve.*

Let $AC = t$, $CD = c$, $AP = p$, $PR = q$, $PQ = x$; then $QB^2 = \frac{cc}{tt} \times \overline{pq+qx-px-xx}$, and $PB^2 = \frac{cc}{tt} \times \overline{pq+qx-px-xx} + xx = m^2$: In Fluxions $\frac{ccq\dot{x} - ccp\dot{x} - 2ccx\dot{x}}{tt} + 2x\dot{x} = 0$; reduced $x = \frac{cc \times \overline{p-q}}{2 \times \overline{tt-cc}} = \frac{cc}{tt-cc} \times CP$.

Note, If CP be greater than $\frac{tt-cc}{t}$, then x will be greater than PR, which the Nature of the Queſtion will not admit.

Ex.

Ex. 7.

FIG. 22.

To draw the Line EF to touch the Angle C of the Rectangle ABCD, ſo that the Part, contain'd between the Sides AB, AD produced, may be a Minimum.

Let $AD=a$, $AB=b$, $EB=x$, $EA=b+x$, then $AF=\frac{ab+ax}{x}$, and $EF=\sqrt{\overline{b+x}^2+\frac{aa}{xx}\times\overline{b+x}^2}$ $=m$, or $bb+2bx+xx+\frac{aabb}{xx}+\frac{2a^2b}{x}+aa=m^2$: This in Fluxions is $2b\dot{x}+2x\dot{x}-\frac{2aabb\dot{x}}{x^3}-\frac{2aab\dot{x}}{xx}$ $=0$; reduced is $x^4+bx^3-aabx-aabb=0$, divided by $x+b=0$, one of the Roots, and then $x^3-aab=0$, and $x=\sqrt[3]{aab}$, or from the other Equation $x=-b$.

Or thus:

By ſimilar Triangles $x:a::b:\frac{ba}{x}=DF$, then $EF=\sqrt{aa+xx}+\sqrt{bb+\frac{aabb}{xx}}=m$: In Fluxions $\frac{x\dot{x}}{\sqrt{aa+xx}}-\frac{aabb\dot{x}}{x^3\sqrt{bb+\frac{aabb}{xx}}}=0$; reduced $x^3=aab$.

Ex. 8.

To find a Cone of the greateſt Solidity under a given Convex Surface and Baſe b.

Let the Diameter of the Baſe $=x$, Side $=v$, $=3.1416$, then the Surface $=\frac{cxx}{4}+\frac{cxv}{2}=b$, and

Solidity

FIG. 22. Solidity $= \frac{cxx}{12}\sqrt{vv - \frac{1}{4}xx} = m$; and expunging v, $\frac{cxx}{12}\sqrt{\frac{4bb}{ccxx} - \frac{2b}{c}} = m$, or $4bbxx - 2bcx^4 = 144m^2$: In Fluxions, $8bbx\dot{x} - 8bcx^3\dot{x} = 0$; reduced $x^2 = \frac{b}{c}$; hence $x = \sqrt{\frac{b}{c}}$, $v = \frac{3}{2}\sqrt{\frac{b}{c}}$, and Height $= \sqrt{\frac{2b}{c}}$, and the Height, Baſe, and Side will be as $\sqrt{2}$, 1 and $1\frac{1}{2}$. And the ſame would be true if a Cone of a given Solidity, under the leaſt Surface was required.

Ex. 9.

To find y *and* x *ſuch, that* $ay^3 - y^2x^2 + x^4$ *may be a Minimum.*

This Equation in Fluxions is $3ay^2\dot{y} - 2x^2y\dot{y} - 2y^2x\dot{x} + 4x^3\dot{x} = 0$: And comparing the homologous Terms, $3ay^2\dot{y} - 2x^2y\dot{y} = 0$, and $4x^3\dot{x} - 2y^2x\dot{x} = 0$; whence $3ay^2 = 2x^2y$, and $2x^2 = yy$, and therefore $3ay = yy$, and thence $y = 0$, or $y = 3a$; hence $x^2 = 0$, or $x^2 = \frac{9}{2}aa$.

Note, if y be given, the Quantity $ay^3 - y^2x^2 + x^4$ has a Minimum, but the Maximum is infinite; and if x be given, it has a Minimum, but the Maximum is alſo infinite: Therefore if neither be given, it has a Minimum, but no Maximum.

Ex. 10.

To find $xy^2u^3z^4$ *a Maximum ſo that* $x+y+u+z=b$.

Here $xy^2u^3z^4 = m$, and expunging x, $y^2u^3z^4 \times \overline{b-y-u-z} = m$, or $b-y-u-z = \frac{m}{y^2u^3z^4}$: In Fluxions, $-\dot{y} - \dot{u} - \dot{z} = -\frac{2m\dot{y}}{y^3u^3z^4} - \frac{3m\dot{u}}{y^2u^4z^4} -$

$4m\dot{z}$

$\frac{4m\dot{z}}{y^2u^3z^5}$; and collecting ſeparately the homologous FIG.
Quantities, $\dot{y} = \frac{2m\dot{y}}{y^3u^3z^4}$, $\dot{u} = \frac{3m\dot{u}}{y^2u^4z^4}$, $\dot{z} = \frac{4m\dot{z}}{y^2u^3z^5}$; whence $\frac{m}{y^2u^3z^4} = \frac{y}{2} = \frac{u}{3} = \frac{z}{4} = b-y-u-z$; therefore $u = \frac{3}{2}y$, $z = \frac{4}{3}u = 2y$, and therefore $\frac{1}{2}y = b - \frac{9}{2}y$, or $y = \frac{1}{5}b$; hence $x = \frac{1}{10}b$, $y = \frac{2}{10}b$, $u = \frac{3}{10}b$, $z = \frac{4}{10}b$.

Ex. 11.

To find a Trapezoid ABCD of a given Area b, whoſe two Sides and Baſe AB+BC+CD ſhall be the leaſt poſſible. 23.

Let $BC = x$, $BA = y$, $CD = u$, perpendicular BE or $CF = z$. Then $y+x+u = m$, and $z\sqrt{yy-zz} + z\sqrt{uu-zz} + 2zx = 2b$, or (dividing by z) $\sqrt{yy-zz} + \sqrt{uu-zz} + 2x = \frac{2b}{z}$. Put theſe Equations into Fluxions, and $\dot{y} + \dot{x} + \dot{u} = 0$, and $\frac{y\dot{y} - z\dot{z}}{\sqrt{yy-zz}} + \frac{u\dot{u} - z\dot{z}}{\sqrt{uu-zz}} + 2\dot{x} = \frac{-2b\dot{z}}{zz}$; and expunging $\dot{x}$, $\frac{y\dot{y} - z\dot{z}}{\sqrt{yy-zz}} + \frac{u\dot{u} - z\dot{z}}{\sqrt{uu-zz}} - 2\dot{y} - 2\dot{u} = \frac{-2b\dot{z}}{zz}$; whence $\frac{y\dot{y}}{\sqrt{yy-zz}} - 2\dot{y} = 0$, and $4z^2 = 3y^2$. Alſo $\frac{u\dot{u}}{\sqrt{uu-zz}} - 2\dot{u} = 0$, and thence $4z^2 = 3uu$. Therefore $3y^2 = 3u^2$, and $y = u$. Laſtly $\frac{-z\dot{z}}{\sqrt{yy-zz}} - \frac{z\dot{z}}{\sqrt{uu-zz}} = \frac{-2b\dot{z}}{zz}$,

that

FIG. 23. that is $\frac{z^3}{\sqrt{yy-zz}} + \frac{z^3}{\sqrt{uu-zz}} = 2b$; and expunging z and u, $\frac{z^3}{\frac{1}{2}y} + \frac{z^3}{\frac{1}{2}u} = \frac{4z^3}{y} = \frac{3}{2}y^2\sqrt{3} = 2b$, and $y = 2\sqrt[4]{\frac{bb}{27}} = \frac{2}{3}\sqrt[4]{3bb} = u$, and $x = \frac{b}{z} - \frac{1}{2}\sqrt{yy-zz} - \frac{1}{2}\sqrt{uu-zz} = \frac{2}{3}\sqrt[4]{3bb}$; and $z = \sqrt[4]{\frac{bb}{3}}$. Hence $AB = 2AE = 2FD$, and $AB = BC = CD$: And the Figure is inſcribed in a Semicircle, whoſe Diameter is AD.

Ex. 12.

24. *Given the Velocity of a Projectile, and its Height AB above the Horizon: To find the Angle of Elevation LAG, to throw it to the greateſt Diſtance poſſible, on the horizontal Plane BC.*

Let $s = AD$ the Space deſcribed in a given Time, $d = DE$ the Space deſcribed by a falling Body in the ſame Time; $b = AB$; and let AG be the Line of Projection, AL parallel to BC, $GL = x$. Then by the Laws of falling Bodies, $d : b+x :: ss : \frac{bss+ssx}{d} = AG^2$. Therefore $AL = \sqrt{\frac{bss+ssx}{d} - xx} = BC = m$, or $bss + ssx - dxx = dmm$; in Fluxions, $ss\dot{x} - 2dx\dot{x} = 0$, whence $x = \frac{ss}{2d}$. Therefore $AG = \frac{s}{2d}\sqrt{4bd + 2ss}$, $AL = \frac{s}{2d}\sqrt{4bd + ss}$, $AC = b + \frac{ss}{2d} = CG$; whence the Angle FAC is biſſected by the Line of Elevation or Direction AG; and $\sqrt{4bd + ss} : s :: (AL : GL ::)$ Rad. : Tangent of the Angle of Elevation LAG.

Ex. 13.

If the Weight w be given, and the Radii of the Wheel and Axel BA, AC; to find the weight y, to be raiſed by the Deſcent of w, ſo that y may receive the greateſt Motion poſſible in a given Time. FIG. 231.

Let $AC = a$, $AB = b$, and put $p + q = w$, ſo that the Part p may juſt balance y upon the Wheel, and therefore $bp = ay$. Let $2h$ = Velocity gain'd by Gravity in one Second, and v = Velocity gain'd by y in its Aſcent, in the ſame Time; and then $\frac{b}{a}v$ = Velocity of w; and the Motion which would be generated in q by its Gravity will be $= 2hq$. But ſince the Motion of the Bodies w, y, is generated by the ſame Gravity of q, therefore that Motion will be equal to the former; for when the Force is given, the Motion generated, in a given Time, will be given. But $vy + \frac{b}{a}vw$ = Motion generated in w and y, therefore $vy + \frac{b}{a}vw = 2hq$. But $q = w - p = w - \frac{ay}{b}$; whence $v = \frac{2hq}{\frac{b}{a}w + y} = \frac{2h \times \overline{w - \frac{a}{b}y}}{\frac{b}{a}w + y} = \frac{2ah \times \overline{bw - ay}}{b \times \overline{bw + ay}}$; and $vy = \frac{2ahy \times \overline{bw - ay}}{b \times \overline{bw + ay}} = m$, by the Queſtion, or $\frac{bwy - ayy}{bw + ay} = \frac{bm}{2ah}$. In Fluxions, $\overline{bw\dot{y} - 2ay\dot{y}} \times \overline{bw + ay} - a\dot{y} \times \overline{bwy - ayy} = 0$, and $bbww - 2abwy - aayy = 0$; which re-

FIG. duced is $y = \frac{\surd 2 - 1}{1} \times \frac{bw}{a} = \frac{.414bw}{a}$. And hence is found $v = \frac{2ha}{b} \times : \surd 2 - 1 :$

And if $b = a$, then $y = .414w$.

SCHOLIUM.

If the Brachium a was ſought, all the reſt being given, we ſhould come to the ſame Concluſion, that is, $a = \frac{.414bw}{y}$.

Ex. 14.

232. *The Arch AD being given, to divide it into three Parts, AB, BC, CD; ſo that the Product of any Powers of the Sines* $BE^m \times CI^n \times DO^r$, *may be the greateſt poſſible.*

Let Radius $RA = 1$, Arches $AB = A$, $BC = B$, $CD = C$, and $AD = g$; Sines $BE = x$, $CI = y$, $DO = s$; Coſines $RE = X$, $RI = Y$, $RO = S$; $Bq = \dot{A}$, $Bp = \dot{x}$, and let $x^m y^n s^r =$ Max. by ſimilar Triangles $X\,(RE)$: $1\ (RB) :: \dot{x}\,(Bp) : \dot{A}\,(Bq) = \frac{\dot{x}}{X}$; likewiſe $\dot{B} = \frac{\dot{y}}{Y}$, and $\dot{C} = \frac{\dot{s}}{S}$. And ſince $A + B + C = g$, therefore $\dot{A} + \dot{B} + \dot{C} = 0$, or $\frac{\dot{x}}{X} + \frac{\dot{y}}{Y} + \frac{\dot{s}}{S} = 0$. And ſince $x^m y^n s^r =$ Max. therefore $mx^{m-1}\dot{x}y^n s^r + ny^{n-1}\dot{y}x^m s^r + rs^{r-1}\dot{s}x^m y^n = 0$, or $\frac{m\dot{x}}{x} + \frac{n\dot{y}}{y} + \frac{r\dot{s}}{s} = 0$. From theſe two Equations we get $-\dot{s} = S \times : \frac{\dot{x}}{X} + \frac{\dot{y}}{Y}$; and $-\dot{s} = \frac{s}{r} \times : \frac{m\dot{x}}{x} + \frac{n\dot{y}}{y}$: therefore $\frac{S\dot{x}}{X} + \frac{S\dot{y}}{Y} = \frac{sm\dot{x}}{rx} + \frac{sn\dot{y}}{ry}$. Hence (according to Art. 1 of the Rule) we ſhall have $\frac{S\dot{x}}{X} = \frac{sm\dot{x}}{rx}$, and $\frac{S\dot{y}}{Y} = \frac{sn\dot{y}}{ry}$; therefore $\frac{rx}{X} = \frac{ms}{S}$, and $\frac{ry}{Y} = \frac{ns}{S}$. Hence $\frac{x}{X} : \frac{s}{S} :: m : r$, and $\frac{s}{S} : \frac{y}{Y} :: r : n$; and *ex e quo*, $\frac{x}{X} : \frac{y}{Y} :: m : n$. But by Trigonometry, $\frac{x}{X}$, $\frac{y}{Y}$ and $\frac{s}{S}$ are the Tangents of the Arches A, B, C; therefore the Tangents of the Arches A, B, C, are reſpectively as the Indices of the Powers m, n, r. And the ſame holds true for any Number of Arches whatever.

FIG. 232.

Cor. 1. *If there be only two Arches* A, B; then $m+n : m-n ::$ Tan. $A + T.B : T.A - T.B ::$ (by Trigonometry) $S.\overline{A+B} : S.A-B$; that is $m+n : m-n :: S.$ Sum of the Arches A, B : $S.$ of the Difference; whence the Arches will be known.

Or thus. Let Tangent of $A = mv$, of $B = nv$, of $A+B = t$. Then by Trigonometry $\frac{\overline{m+n} \times v}{1 - mnvv} = t$, a quadratic Equation for v.

Cor. 2. *If there be three Arches* A, B, C; let their Tangents be mv, nv, rv, and Tan. Sum $= t$; Tan. $\overline{A+B} = z$. Then by Trigonometry $\frac{\overline{m+n} \times v}{1 - mnvv} = z$, and $\frac{z + rv}{1 - zrv} = t$; reduced $\frac{\overline{m+n+r}.v - mnrv^3}{1 - \overline{mn+mr+nr}.vv} = t$, a cubic Equation for v.

And if there be 4 Arches, you'll have a 4th Power; if 5 Arches, a 5th Power of v, &c.

Ex. 15.

FIG. *To find* x^s *a Maximum, putting* $s = \frac{1}{x^x}$.

Let $X =$ Hyp. Log. of x. Then ſince x^s is a *Maximum*, its Log. (by Art. 1.) is a *Maximum*; that is, sX or $\frac{X}{x^x}$ is a *Maximum:* and the Log. of $\frac{X}{x^x}$ is alſo a *Maximum*; that is, Log. $X —$ Log. x^x or Log. $X — xX$ is a *Max.* and its Fluxion, $\frac{\dot{X}}{X} — x\dot{X} — X\dot{x} = 0$. But $\dot{X} = \frac{\dot{x}}{x}$, (ſee the next Prob.); therefore $\frac{\dot{x}}{xX} — \dot{x} — X\dot{x} = 0$; whence $X^2 x + Xx = 1$.

Let $x = 1 + v$; then $X = v — \frac{v^2}{2} + \frac{v^3}{3} — \frac{v^4}{4}$ &c. (by Example 3, of the next Prob.); therefore $\overline{v — \frac{vv}{2} + \frac{v^3}{3} \ \&c.}^{\,2} + \overline{v — \frac{vv}{2} + \frac{v^3}{3} \ \&c.} \times : \overline{1+v} : = 1$. That is, $v + \frac{3}{2}v^2 — \frac{1}{6}v^3$ &c. $= 1$. Whence, by Reverſion, $v = .56$, and $x = 1.56$.

But becauſe this does not converge faſt, put $n = 1.56$, and $n + v = x$; $l =$ Hyp. Log. n ($= 2.302585 \times$ vulgar Log. n), then will $X = l + \frac{v}{n} — \frac{vv}{2nn}$ + &c. whence we ſhall have $\overline{l + \frac{v}{n} — \frac{vv}{2nn}}^{\,2}$

+

$+l+\frac{v}{n}-\frac{vv}{2nn}:\times\overline{n+v}:=1.$ That is, FIG.

$$\overline{l+1}\times nl+\overline{\overline{l+1}^2+l}\times v+\frac{l+\frac{3}{2}}{n}vv,\ \&c.=1.$$

In Numbers.

$1.0021921+2.5318022v+1.246593vv$, &c. $=1.$
Or $2.031v+vv=-.0017586.$
Whence $v=-{,}0008661$; and $n+v$ or $x=1.5591339.$

Otherwise thus.

Let $l+p=X$; $n=$ Number of the Hyp. Logarithm l, as before. Then x (or the Number of the Hyp. Logarithm $l+s$ or of X is) $=n\times:1+p+\frac{1}{2}pp+\frac{1}{6}p^3$ &c. (by Schol. 2. of the next Prob.) Whence the Equation $\overline{XX+X}\times x=1$, becomes $\overline{l+p}^2+l+p\times$ *into* $n+np+\frac{1}{2}npp$ &c. $=1.$

That is,

$$\overline{l+1}\times l+\left.\begin{matrix}\overline{l+1}^2\\+l\end{matrix}\right\}p\;\left.\begin{matrix}+\frac{1}{2}ll\\+2\frac{1}{2}l\\+2\end{matrix}\right\}p^2:\times n=1.$$

In Numbers.

$1.002192+3.949611p+5.008515pp$ &c. $=1.$
Or $.78858p+pp=-{,}00043768.$
Whence $p=-{,}0005549$, and $l+p=.4441309$ $=$ Hyp. Log. x; or $\overline{l+p}\times{,}4342945=$ vulgar Log. of x (See Cor. to Ex. 8th of the next Prob.) $=0.1928836$; whence $x=1.559134.$

Note x^s or $x^{\frac{1}{x^x}}$ has no Minimum.

Here follow ſome Examples of finding the Nature of Curves, that require ſome *Maximum* or *Minimum.*

Ex. 16.

230. *To determine the Nature of the Curve EFD, ſo that the Length of the Arch AD being given, the Area ABD ſhall be a Maximum.*

Let *HFGI* be an infinitely ſmall given Part of the Area, and *C* an intermediate Point (according to the Rule Art. 3.) and draw the Ordinates *FH*, *GI*, and *CP*; and *Fm*, *Cn* parallel to *AB*; and ſuppoſe $Cm = Gn$.

Put the given Quantities $FH = p$, $CP = q$, Cm or $Gn = n$; and the variable ones $Fm = v$, $Cn = s$.

Then by the Nature of the Problem, $FC + CG =$ a given Quantity $= \sqrt{vv+nn} + \sqrt{ss+nn}$: Therefore $\frac{v\dot{v}}{\sqrt{vv+nn}} + \frac{s\dot{s}}{\sqrt{ss+nn}} = 0$. And becauſe the Area $pv + qs$ is a Max. $= m$, therefore $p\dot{v} + q\dot{s} = 0$, or $p\dot{v} = -q\dot{s}$, whence $\frac{v\dot{v}}{\sqrt{vv+nn}} = \frac{-s\dot{s}}{\sqrt{ss+nn}} = \frac{sp\dot{v}}{q\sqrt{ss+nn}}$; that is (making the Ordinates alike affected) $\frac{v}{p\sqrt{vv+nn}} = \frac{s}{q\sqrt{ss+nn}}$, or $\frac{Fm}{FH \times FC} = \frac{Cn}{CP \times CG}$; conſequently (any one as) $\frac{Fm}{FH \times FC}$ is an invariable Quantity. Therefore if $AH = x$, $HF = y$, $EF = z$, then $\frac{\dot{x}}{y\dot{z}} = \frac{1}{a}$, or $a\dot{x} = y\dot{z}$, for the Nature of the Curve, which will be a Circle.

If

FIG. 230. *If the Length of the Arch be given*, and the Solid to be a Maximum; then will $p^2v + q^2s = m$; whence $p^2\dot{v} = -q^2\dot{s}$; which gives $\frac{v}{pp\sqrt{vv+nn}}$ or $\frac{\dot{x}}{yy\dot{z}}$ a given Quantity, or $aa\dot{x} = yy\dot{z}$.

If the Surface AD be given, to have the Solid a Maximum. Then you'll have $p\sqrt{vv+nn} + q\sqrt{ss+nn}$ = a given Quantity; and $ppv + qqs = m$; which gives $\frac{v}{p\sqrt{vv+nn}}$ or $\frac{\dot{x}}{y\dot{z}}$ a given Quantity, that is $a\dot{x} = y\dot{z}$, as in the firſt. The ſame Equation will be found, *if the Solid be given*, and the Surface be a Minimum.

Ex. 17.

233. *To find the Nature of the Curve ERD, which generates the Solid of leaſt Reſiſtance; whoſe Length and greateſt Diameter are given.*

Let *CP*, an infinitely ſmall Part of the Axis, be given; draw the Ordinates *CR*, *PT*; and the intermediate one *QF*; and let $PT = p$, $QF = q$, $Tm = v$, $Fn = s$, $Fm = Rn = n$.

Then by Mechanics, the Force of the Fluid againſt the Part *TF* of the Surface, will be as the Cube of the Sine of Incidence, and magnitude of the Surface, that is as $\frac{n^3}{TF^3} \times p \times FT$, or $\frac{pn^3}{FT^2}$. Therefore the Force of the Fluid upon the two Parts *TF*, *FR*; will be $\frac{n^3p}{n^2+v^2} + \frac{n^3q}{nn+ss} = m$, a Minimum. Therefore $\frac{-2n^3pv\dot{v}}{\overline{nn+vv}^2} - \frac{2n^3qs\dot{s}}{\overline{nn+ss}^2} = 0$. And ince $s + v =$ a given Quantity, therefore $\dot{v} = -\dot{s}$, whence

FIG. 233. whence $\frac{n^3 p v \dot v}{\overline{nn+vv}^2} = \frac{n^3 q s \dot v}{\overline{nn+ss}^2}$, or $\frac{n^3 p v}{TF^4} =$ a given Quantity; that is, putting $p = y$, $v = \dot x$, $\frac{y \dot y^3 \dot x}{\dot z^4} = a$, or $y \dot y^3 \dot x = a \dot z^4$.

When the Area of the generating Plane ABED, and the greateſt Diameter BE are given; then $pv + qs =$ a given Quantity, and $p \dot v = -q \dot s$; which gives $\frac{n^3 v}{\overline{nn+vv}^2} = \frac{n^3 s}{\overline{nn+ss}^2}$, or $\frac{\dot y^3 \dot x}{\dot z^4} = a$.

If its Bulk AE, and greateſt Diameter be given; then $p^2 v + qqs =$ a given Quantity, or $p^2 \dot v = -q^2 \dot s$, which gives $\frac{n^3 v}{p \times TF^4} = \frac{n^3 s}{q \times FR^4}$, or $\frac{\dot y^3 \dot x}{y \dot z^4} = a$.

Ex. 18.

234. *To find the Nature of the Curve ADK, wherein a Body will deſcend from A to K in the ſhorteſt Time; the Points A, K being given.*

Suppoſe *BE* an infinitely ſmall given Part of the Axis, and draw the Ordinates *BD*, *EH*, and the middle one *CM*; and let the invariable Lines be $BD = p$, $CM = q$, $oM = n = gH$; and the variable ones, $BC = v$, $CE = s$; then the Velocity at *D* will be as $\sqrt{BD}$, and at *M* as $\sqrt{CM}$; and the Times being as the Spaces directly and Velocities reciprocally, therefore $\frac{DM}{\sqrt{BD}}$ will be as the Time of deſcribing *DM*, and $\frac{MH}{\sqrt{CM}}$ as the Time in *MH*; therefore $\frac{\sqrt{vv+nn}}{\sqrt{p}} + \frac{\sqrt{ss+nn}}{\sqrt{q}} = m$, alſo $v + s =$ a given Quantity;

whence $\frac{v\dot{v}}{\sqrt{p}\sqrt{vv+nn}} + \frac{s\dot{s}}{\sqrt{q}\sqrt{ss+nn}} = 0$, and FIG. 234. $\dot{v} + \dot{s} = 0$; therefore we have $\frac{v}{\sqrt{p}\,\sqrt{vv+nn}} = \frac{s}{\sqrt{q}\sqrt{ss+nn}}$; that is (putting $AB = x$, $BD = y$) $\frac{\dot{x}}{\dot{z}\sqrt{y}} = \frac{1}{\sqrt{a}}$, a given Quantity; whence $\dot{x}\sqrt{a} = \dot{z}\sqrt{y}$, or $a\dot{x}^2 = y\dot{z}^2 = y\dot{x}^2 + y\dot{y}^2$, and $\dot{x}\sqrt{a-y} = \dot{y}\sqrt{y}$, which denotes a Cycloid.

If the Velocity at D be as any Power or Sums of Powers of BD, which call R; it will appear the ſame Way, that $\frac{\dot{x}}{R\dot{z}} =$ a given Quantity.

If the Area ARK be given; then $pv + qs$ is given, or $p\dot{v} = -q\dot{s}$, whence $\frac{v\dot{v}}{p^{\frac{3}{2}} \times DM} = \frac{s\dot{v}}{q^{\frac{3}{2}} \times MH}$, or $\frac{\dot{x}}{y^{\frac{3}{2}}\dot{z}} = \frac{1}{a^{\frac{3}{2}}}$; and if the Velocity at D be as BD^m, then $\frac{\dot{x}}{y^{m+1}\dot{z}} = \frac{1}{a^{m+1}}$.

Ex. 19.

Given CA, CT and the Length of the Curve ADT; to find the Nature of it, ſo that the *Area ACT may be a Maximum.* 235.

Let the infinitely ſmall Parts of the Area BCE, and of the Curve BE be given: and draw the mean Ordinate CD, and put $CD = p$, $CE = q$; $Bo = v$, $Dg = s$, Do or $Eg = n$. Then $\sqrt{vv+nn} + \sqrt{ss+nn}$ $=$ a given Quantity, and $\frac{pv+qs}{2} = m$, a Maximum.

FIG. 235. Therefore $\frac{v\dot{v}}{\sqrt{vv+nn}} + \frac{s\dot{s}}{\sqrt{ss+nn}} = 0$, and $p\dot{v} = -q\dot{s}$; whence $\frac{v\dot{v}}{\sqrt{vv+nn}} = \frac{sp\dot{v}}{q\sqrt{ss+nn}}$; or $\frac{v}{p \times BD} = \frac{s}{q \times DE}$: Put $CB = y$, $Bo = \dot{x}$, $Do = \dot{y}$; then $\frac{\dot{x}}{y\dot{z}} = \frac{1}{a}$, a given Quantity, or $a\dot{x} = y\dot{z}$, an Equation to the Circle.

Ex. 20.

To find the Nature of the Curve ADT, in which a Body will move from T to A in the leaſt Time poſſible; the Velocity at any Point E being as CE^m, the m^{th} Power of the Diſtance.

Take *ECB* an infinitely ſmall given Angle, and draw the mean Ordinate *CD*; and put $CB = p$, $CD = q$, $Bo = v$, $Dg = s$, *Do* or $Eg = n$. Then the Time being as the Space directly and Velocity reciprocally, we ſhall have $\frac{\sqrt{vv+nn}}{p^m} + \frac{\sqrt{ss+nn}}{q^m} = m$. Whence $\frac{v\dot{v}}{p^m\sqrt{vv+nn}} + \frac{s\dot{s}}{q^m\sqrt{ss+nn}} = 0$, and ſince the Angle *ECB* is given, if Rad. $= 1$, then $\frac{v}{p} = \angle BCD$, $\frac{s}{q} = \angle DCE$, and $\frac{v}{p} + \frac{s}{q} =$ a given Quantity, and $\frac{\dot{v}}{p} = \frac{-\dot{s}}{q}$; therefore $\frac{v\dot{v}}{p^m\sqrt{v^2+n^2}} = \frac{-s\dot{s}}{q^m\sqrt{ss+nn}} = \frac{sq\dot{v}}{pq^m\sqrt{ss+nn}}$, or

FIG. 235.

or $\frac{v}{p^{m-1}\times BD} = \frac{s}{q^{m-1}\times DE}$; that is $\frac{\dot{x}}{y^{m-1}\dot{z}}$ $= \frac{1}{a^{m-1}}$, a given Quantity.

If $m=0$, AD is a right Line, where a is the Perpendicular from C.

If $m=1$, then $\frac{\dot{x}}{\dot{z}}$ is a conſtant Quantity, and the Curve is a Log. Spiral.

If $m=2$, then $a\dot{x}=y\dot{z}$, the Equation of a Circle.

If $R=$ *any Sums of the Powers of* y, repreſenting the Velocity; then will $\frac{y\dot{x}}{R\dot{z}}=$ a conſtant Quantity as before.

Here follow ſome Examples of Curves, where more Conditions are concerned in the Queſtion.

Ex. 21.

To determine the Nature of the Curve EFD, *ſo that the Points* E, D *being given, as alſo the Length of the Arch* ED; *the Area* ABD *ſhall be a Maximum.* 230.

Let $AH=x$, $HF=y$, $EF=z$, then ſince the Points E, D, are given, the Length of the Axis AB is given, therefore the Axis $AB=$ F: $a\dot{x}$; alſo the Curve ED $=$ F: $b\dot{z}$; and the Area of the Figure $=$ F: $cy\dot{x}$. Therefore collecting the Quantities that are affected with $\dot{z}$, and alſo thoſe that are affected with $\dot{x}$; we ſhall have F: $A\dot{z}$ — F: $B\dot{x}$ $=$ F: $b\dot{z}$ — F: $\overline{a+cy}\times\dot{x}$; whence $A=b$, $B=a+cy$, therefore by the Rule Art. 4. $A\dot{x}=B\dot{z}$ becomes $b\dot{x}=a\dot{z}+cy\dot{z}$, for the Nature of the Curve, which denotes the Arch of a Circle.

If the Solid generated about AB, is required to be a FIG. 230. *Maximum*; then the Solid $=$ F: $dyy\dot{x}$; and $A=b$, $B=a+dyy$, and $b\dot{x}=a\dot{z}+dyy\dot{z}$ is the Nature of the Curve.

If the Surface generated by ED is given, to have the Solid a Maximum; then the Surface $=$ F: $gy\dot{z}$, whence $A=b+gy$, $B=dyy$; and $gy\dot{x}=a\dot{z}+dyy\dot{z}$, for the Nature of the Curve. Here the Coefficients a, b, c, d, g are conſtant Quantities, affirmative or negative, to be determined by the Circumſtances of the Queſtion.

Ex. 22.

233. *To find the Nature of the Curve ERD, which generates the Solid of leaſt Reſiſtance moving in a Fluid; whoſe Length, greateſt Diameter, and ſolid Content is given.*

Let $AP=x$, $PT=y$, $DT=z$; then Axis $AB=$ F: $a\dot{x}$, and Solidity $=$ F: $byy\dot{x}$, then by the Reaſoning in Ex. 16. the Force of the Fluid upon the Annulus $PTFQ$ is $\frac{ry\dot{y}^3}{\dot{z}^3}\times\dot{z}$, which is a Minimum; and the whole Reſiſtance $=$ F: $\frac{ry\dot{y}^3}{\dot{z}^3}\times\dot{z}$. And collecting the Quantities affected with $\dot{z}$, and alſo thoſe affected with $\dot{x}$, and we have F: $A\dot{z}$ — F: $B\dot{x}=$ F: $\frac{ry\dot{y}^3}{\dot{z}^3}\times\dot{z}$ — F: $\overline{a\dot{x}+byy\dot{x}}$; whence $A=\frac{ry\dot{y}^3}{\dot{z}^3}$, $B=a+byy$. Therefore $(A\dot{x}=B\dot{z})$ $\frac{ry\dot{y}^3}{\dot{z}^3}\dot{x}=a\dot{z}+byy\dot{z}$, for the Nature of the Curve.

If inſtead of the Solidity, the Surface is given; then the Surface $=$ F: $cy\dot{z}$. Therefore $A=\frac{ry\dot{y}^3}{\dot{z}^3}+cy$, and $B=a$. Then $(A\dot{x}=B\dot{z})$ $\frac{ry\dot{y}^3}{\dot{z}^3}\dot{x}+cy\dot{x}=a\dot{z}$, for the Nature of the Curve DTE.

If both the Surface and Solidity be given; then $A = \frac{ry\dot{y}^3}{\dot{z}^3} + cy$, $B = a + byy$. And $(A\dot{x} = B\dot{z})$ $\frac{ry\dot{y}^3}{\dot{z}^3}\dot{x} + cy\dot{x} = a\dot{z} + byy\dot{z}$, for the Nature of the Curve.

Ex. 23.

To find the Nature of the Curve ADK, wherein a Body will deſcend from A to K, in the ſhorteſt Time; the Points A, K being given, and the Length of the Curve ADK. 234.

Let $AB = x$, $BD = y$, $AD = z$. The Velocity at D is as $\sqrt{y}$, and the Time of deſcribing DM or $\dot{z}$, is as the Space divided by the Velocity, or as $\frac{\dot{z}}{\sqrt{y}}$, and the whole Time of deſcribing $ADK =$ F: $\frac{a\dot{z}}{\sqrt{y}}$, and the Axis $AR =$ F: $b\dot{x}$, alſo the Curve $ADK =$ F: $c\dot{z}$. Then F: $A\dot{z}$ — F: $B\dot{x} =$ F: $\frac{a}{\sqrt{y}} + c \times \dot{z}$ — F: $b\dot{x}$. Whence $A = \frac{a}{\sqrt{y}} + c$, $B = b$, and $(A\dot{x} = B\dot{z})$ $c\dot{x} + \frac{a\dot{x}}{\sqrt{y}} = b\dot{z}$, for the Nature of the Curve of quickeſt Deſcent.

If the Velocity at D be ſuppoſed to be as y^n, the Equation of the Curve will come out $c\dot{x} + \frac{a\dot{x}}{y^n} = b\dot{z}$.

Ex. 24.

The Length of the Curve ADG, and the Baſe AG being given, to find the Nature of it, ſo that its Center of Gravity deſcends lower than that of any other Line of the ſame Length, when the Points A, G are fixt. 236.

Let $AB = x$, $BD = y$, $AD = z$; then the Baſe $AG =$ F: $a\dot{x}$, the Curve $ADG =$ F: $b\dot{z}$, and the Diſtance of

FIG. 236.

of the Center of Gravity from AG is $= F: cy\dot{z}$, by Prob. XVII. following. Hence $F: A\dot{z} - F: B\dot{x} = F: \overline{b+cy} \times \dot{z} - F: a\dot{x}$, and $A = b+cy$, $B = a$; and $(A\dot{x} = B\dot{z})$ $\overline{b+cy} \times \dot{x} = a\dot{z}$, for the Equation of the Curve. In C, where $\dot{x} = \dot{z}$, let $y = d$, and then $b+cd = a$. Let $CE = v$, $y = d - v$, and $\overline{b+cd-cv} \times \dot{x}$ or $\overline{a-cv} \times \dot{x} = a\dot{z}$, and if $c = -1$, then $\dot{x} = \frac{a\dot{z}}{a+v}$, an Equation to the Catenary, by Ex. 12. Prop. XIII.

PROB. II.

To find the Logarithms of Numbers.

Logarithms are a certain Set of Numbers, ſo contrived to anſwer to a Set of Numbers in their natural Order, that the Sum of the Logarithms of any two Numbers ſhall be the Logarithm of the Product of theſe Numbers.

Hence therefore, ſince $1 \times 1 = 1$, the Log. of $1 +$ Log. $1 =$ Log. 1, that is the Log. $1 = 0$.

Alſo let $AB = C$, then Log. $A +$ Log. $B =$ Log. C; and thence Log. A or Log. $\frac{C}{B} =$ Log. $C -$ Log. B.

Again, Log. $A^2 =$ Log. $A +$ Log. $A = 2$ Log. A; And Log. $A^3 =$ Log. $A +$ Log. $A +$ Log. $A = 3$ Log. A; And for the ſame Reaſon Log. $A^n = n \times$ Log. A.

Laſtly, let $AB = 1$, then Log. $A +$ Log. $B =$ Log. $1 = 0$; therefore if the Log. B (or the Log. of a Number greater than 1) be affirmative, the Log. of A (a Number leſs than 1) is negative.

From what has been ſaid it follows, that if there be a Set of Numbers in geometrical Progreſſion proceeding from 1 both Ways *ad infinitum*; and another FIG. Set correſponding thereto, in arithmetic Progreſſion proceeding both Ways from 0; then theſe latter will repreſent the Logarithms of the former, and both will be expreſs'd in the following Form,

Numbers,	$\frac{1}{n^3}$,	$\frac{1}{n^2}$,	$\frac{1}{n}$,	1,	n,	n^2,	n^3,	n^4,	*&c.*
Logarithms,	$-3l$,	$-2l$,	$-l$,	0,	l,	$2l$,	$3l$,	$4l$,	*&c.*

Now let the Number of geometrical Proportionals be increaſed and their Differences decreaſed *ad infinitum*, that the Series may contain all poſſible Numbers; and let the arithmetic Proportionals be in like Manner increaſed; and then the arithmetic Series will lſo contain the Logrithms of all Numbers.

Therefore let n^p be any Number, and pl its Log. hen the Increment of that Number will be $n^{p+1}-n^p$, his Increment divided by the Number itſelf, gives $\frac{n^{p+1}-n^p}{n^p} = n-1$; likewiſe the Increment of the Logarithm is $\overline{p+1}\times l-pl$, that is l. Now ſince —1 and l are the ſame for any Number and its Logarithm, therefore the Increment of the Logarithm vill always be in a given Ratio to the Increment of he Number divided by the Number; therefore if we ut $x =$ Number, and $z =$ its Logarithm, and aſume the given Quantity M, we ſhall have $\dot{z} = M\times$

; and ſince the Fluxions are as the Evaneſcent ncrements, therefore $\dot{z} = M\times\frac{\dot{x}}{x}$, and the Fluent $= M\times$ Fluent of $\frac{\dot{x}}{x}$. Therefore,

To

FIG. *To find the Logarithm of a Number, divide the Fluxion of the Number propoſed, by the Number itſelf; and find the Fluent, which multiply by the conſtant Quantity M, and it will give the Logarithm thereof.*

Note, When $M = 1$, the Logatithms are called the Hyperbolic Logarithms; in which Caſe, the Fluxion of the Logarithm is equal to the Fluxion of the Number divided by the Number. And the Fluxion of the Number is equal to the Number multiplied by the Fluxion of its Logarithm.

EXAMPLE I.

To find the Log. of bx. Here the Fluxion of the Log. is $M\frac{b\dot{x}}{bx} = M\frac{\dot{x}}{x}$; but the Fluent of $M\frac{\dot{x}}{x}$ = Log. x, therefore the Log. bx = Log. x. But when $x = 1$, then Log. bx = Log. b, and Log. $x = 0$; therefore the Fluent corrected is Log. bx — Log. b = Log. x; that is Log. bx = Log. b + Log. x.

This finds the Logarithm of a Product having th Logarithms of the two Factors given.

Ex. 2.

To find the Logarithm of x^n. Here the Fluxion o its Logarithm $= M \times \frac{nx^{n-1}\dot{x}}{x^n} = M \times \frac{n\dot{x}}{x}$; an taking the Fluent, Log. x^n = Fluent of $M \times \frac{n\dot{x}}{x}$ = $n \times$ Log. x, which needs no Correction (becau when $x = 1$, $x^n = 1$, and Log. $x = 0$).

This gives a Rule for finding the Logarithm o any Power or Root of a Quantity, when the Log. o the Quantity is known.

Ex

Ex. 3.

To find the Log. $\frac{n+x}{n}$. The Fluxion of the Log. is $= M\times\frac{\dot{x}}{n+x} = M\times:\frac{\dot{x}}{n} - \frac{x\dot{x}}{nn} + \frac{x^2\dot{x}}{n^3} - \frac{x^3\dot{x}}{n^4}$ + &c. therefore the Log. $\frac{n+x}{n} = M\times:\frac{x}{n} - \frac{x^2}{2n^2} + \frac{x^3}{3n^3} - \frac{x^4}{4n^4}$ + &c. If x be negative the Signs of all the odd Powers muſt be changed.

After the ſame Manner the Log. $\frac{n}{n-x} = M\times: \frac{x}{n} + \frac{x^2}{2n^2} + \frac{x^3}{3n^3} + \frac{x^4}{4n^4}$ &c.

Otherwiſe thus.

To find the Log. of $n+x$. Let $n+x=y$; then the Fluxion of the Log. is $\frac{M\dot{x}}{n+x} = \frac{M\dot{x}}{y}$. And (by Rule 8, Prop. X.) the Fluent or Log. $n+x := M\times: \frac{x}{y} + \frac{x^2}{2y^2} + \frac{x^3}{3y^3} + \frac{x^4}{4y^4}$ &c. and corrected, the Log. $\overline{n+x}$ — Log. $n = M\times: \frac{x}{y} + \frac{x^2}{2y^2} + \frac{x^3}{3y^3}$ &c. And Log. $n+x := $ Log. $n + M\times: \frac{x}{y} + \frac{x^2}{2y^2} + \frac{x^3}{3y^3} + \frac{x^4}{4y^4} + \frac{x^5}{5y^5}$ &c.

Alſo Log: $\frac{n+x}{n} = M\times: \frac{x}{y} + \frac{x^2}{2y^2} + \frac{x^3}{3y^3} + \frac{x^4}{4y^4}$ &c.

COR. *The Log. of* $b \times \frac{n+x}{n} = Log.\ b + M \times : \frac{x}{n} - \frac{x^2}{2n^2} + \frac{x^3}{3n^3} - \frac{x^4}{4n^4}$ *&c.* And *the Log. of* $b \times \frac{n}{n-x} = Log.\ b + M \times : \frac{x}{n} + \frac{x^2}{2n^2} + \frac{x^3}{3n^3} + \frac{x^4}{4n^4}$ *&c.*

SCHOL. Theſe Series will find the Log. of a Fraction, or the Log. of a Product having the Log. of *b*. And it finds the Log. of any large Number $n+x$, having the Log. of *b*, and making $b=n$.

EX. 4.

To find the Log. of $\frac{n+x}{n-x}$. Its Fluxion is $= 2M \times \frac{n\dot{x}}{nn-xx} = 2M \times : \frac{\dot{x}}{n} + \frac{x^2\dot{x}}{n^3} + \frac{x^4\dot{x}}{n^5} + \frac{x^6\dot{x}}{n^6}$ + *&c.* Whence the Fluent or Log. $\frac{n+x}{n-x} = 2M \times : \frac{x}{n} + \frac{x^3}{3n^3} + \frac{x^5}{5n^5} + \frac{x^7}{7n^7}$ + *&c.* Or the Log. $\frac{n+x}{n-x} = \frac{2M \times \frac{x}{n}}{1} + \frac{eA}{3} + \frac{eB}{5} + \frac{eC}{7} + \frac{eD}{9}$ *&c.* where $e = \frac{xx}{nn}$, *A*, *B*, *C*, *&c.* are the Numerators of the preceding Terms, with their Signs.

COR. The Log. of $b \times \frac{n+x}{n-x} = $ Log. $b + \frac{2M \times \frac{x}{n}}{1} + \frac{eA}{3} + \frac{eB}{5} + \frac{eC}{7}$ *&c.*

Schol. The Series in this Example very ſpeedily finds the Log. of a Fraction when x is very ſmall, or n very great.

The Series in this Cor. finds the Log. of a Product, having the Log. b one of the Factors given. Likewiſe it finds the Log. of a large Number $n+x$, having the Log. of b, and making $b=n-x$, and $x=1$ or $\frac{1}{2}$.

Ex. 5.

To find the Log. of n, having the Logarithms of n—x and n+x given.

The Fluxion of the Log. of $\frac{n}{\sqrt{nn-xx}} = \frac{Mx\dot{x}}{nn-xx} = M\times : \frac{x\dot{x}}{nn} + \frac{x^3\dot{x}}{n^4} + \frac{x^5\dot{x}}{n^6} + \frac{x^7\dot{x}}{n^8}$, &c. whence the Log. $\frac{n}{\sqrt{nn-xx}} = M\times : \frac{x^2}{2n^2} + \frac{x^4}{4n^4} + \frac{x^6}{6n^6}$ &c. but Log. $\frac{n}{\sqrt{nn-xx}} = \text{Log. } n - \text{Log. } \sqrt{nn-xx} = \text{Log. } n - \frac{\text{Log}: \overline{n^2-x^2}}{2} = \text{Log. } n - \frac{\text{Log. } \overline{n+x}}{2} - \frac{\text{Log. } \overline{n-x}}{2}$. Therefore $\text{Log. } n = \frac{\text{Log. } \overline{n+x} + \text{Log. } \overline{n-x}}{2} + M\times : \frac{x^2}{2nn} + \frac{x^4}{4n^4} + \frac{x^6}{6n^6} +$ &c. or *the Log.* $n =$

$$\frac{\text{Log. } \overline{n+x} + \text{Log. } \overline{n-x}}{2} + \frac{\frac{1}{2}Me}{1} + \frac{eA}{2} + \frac{eB}{3} + \frac{eC}{4} + \text{\&c.}$$

where $e = \frac{xx}{nn}$; A, B, C, &c. the Numerators of the preceding Terms.

Schol. This Series is very proper for finding the Log. of a large prime Number n, having the Log. of the adjoining Numbers $n-1$ and $n+1$ given, and therefore $x=1$.

Ex. 6.

To find the Log. of n, having the Logarithms of the adjoining Numbers n—1 and n+x given.

Let $2nn-1=y$, then will $\frac{n}{\sqrt{nn-1}} = \frac{\sqrt{\frac{y+1}{2}}}{\sqrt{\frac{y-1}{2}}}$ $= \sqrt{\frac{y+1}{y-1}}$, the Fluxion of the Log. of $\sqrt{\frac{y+1}{y-1}}$ $= \frac{-\dot{y}}{yy} - \frac{\dot{y}}{y^4} - \frac{\dot{y}}{y^6} - \frac{\dot{y}}{y^8}$ &c. $\times M$; therefore the Log. $\sqrt{\frac{y+1}{y-1}} = \frac{1}{y} + \frac{1}{3y^3} + \frac{1}{5y^5}$ $+ \frac{1}{7y^7}$ &c. $\times M$; and when y is infinite, Log. $\sqrt{\frac{y+1}{y-1}}$ $=0$, and $\frac{1}{y} + \frac{1}{3y^3} + \frac{1}{5y^5}$ &c. $=0$; therefore ſtill the Log. $\sqrt{\frac{y+1}{y-1}}$ or Log. $\frac{n}{\sqrt{nn-1}} =$ $M\times : \frac{1}{y} + \frac{1}{3y^3} + \frac{1}{5y^5} + \frac{1}{7y^7}$ &c. Whence

$$\text{Log. } n = \frac{\text{Log. } \overline{n+1} \times \text{Log. } \overline{n-1}}{2} + M\times : \frac{1}{y} + \frac{1}{3y^3} + \frac{1}{5y^5} + \frac{1}{7y^7} + \text{\&c.}$$

Schol. This Series is for the ſame Purpoſe, and converges faſter than that in Example the 5th.

Ex.

Ex. 7.

To find the Logarithm of a large Number n, having the Logarithms of n—x and n+x given.

Put $d = \text{Log.}\,\overline{n+x} - \text{Log.}\,\overline{n-x} = \text{Log.}\,\frac{n+x}{n-x}$.

Then the Log. $n - \text{Log.}\,\overline{n-x} = \text{Log.}\,\frac{n}{n-x} =$

$$\frac{\text{Log.}\,\frac{n+x}{n-x} \times \text{Log.}\,\frac{n}{n-x}}{\text{Log.}\,\frac{n+x}{n-x}} = d \times \frac{\text{Log.}\,\frac{n}{n-x}}{\text{Log.}\,\frac{n+x}{n-x}} =$$

(by Ex. 3 and 4) $d \times \dfrac{M \times : \frac{x}{n} + \frac{x^2}{2nn} + \frac{x^3}{3n^3} + \frac{x^4}{4n^4} \;\&c}{2M \times : \frac{x}{n} + \frac{x^3}{3n^3} + \frac{x^5}{5n^5} + \frac{x^7}{7n^7} \;\&c}$

$= \frac{d}{2} \times : 1 + \frac{x}{2n} + \frac{x^3}{12n^3} + \frac{7x^5}{180n^5} + \&c.$ therefore *Log.* $n =$ *Log.* $\overline{n-x} + \frac{d}{2} \times : 1 + \frac{x}{2n} + \frac{x^3}{12n^3} + \frac{7x^5}{180n^5} + \&c.$ Or thus, ſince $\frac{d}{2} = \frac{\text{Log.}\,\overline{n+x} - \text{Log.}\,\overline{n-x}}{2}$, therefore *Log.* $n =$

$$\frac{\text{Log.}\,\overline{n+x} + \text{Log.}\,\overline{n-x}}{2} + \frac{d}{2} \times : \frac{x}{2n} + \frac{x^3}{12n^3} + \frac{7x^5}{180n^5} + \&c.$$

SCHOL. This Series converges far faſter than either of the former Examples.

Ex. 8.

To find the Hyperbolic Logarithm of 10.

Since $\frac{5^{10} \times 2^{30}}{4^{10} \times 10^9}$ or $\frac{5^{10}}{4^{10}} \times \frac{\overline{1024}^3}{\overline{1000}^3} = 10$; therefore 10 Log.

$10\ \text{Log.}\ \frac{5}{4} + 3\ \text{Log.}\ \frac{1024}{1000} = \text{Log.}\ 10.$ By Ex. 4 $\text{Log.}\ \frac{5}{4} = 2M \times : \frac{1}{9} + \frac{1}{3} \times \frac{1}{9^3} + \frac{1}{5} \times \frac{1}{9^5}$ &c. therefor $10\ \text{Log.}\ \frac{5}{4} = \frac{20}{9} + \frac{Ae}{3} + \frac{Be}{5} + \frac{Ce}{7} + \frac{De}{9} +$ &c (putting $e = \frac{1}{81}$). Likewiſe $3\ \text{Log.}\ \frac{1024}{1000} = \frac{18}{253} + \frac{Ae}{3} + \frac{Be}{5} + \frac{Ce}{7}$ &c. (putting $e = \frac{9}{64009}$). Here $M = 1$, $A =$ firſt Term of each Series, and B, C, D, &c. the Numerators of each preceding Term.

The particular Terms of each Series being found will be as follows.

1) $A = 2,2222222222222222222222222$
3) $Ae = 91449474165523548239598$
5) $Be = 677403512337211468441$
7) $Ce = 5973575946536256335$
9) $De = 57359439815848826$
11) $Ee = 579388280968170$
13) $Fe = 6052489164910$
15) $Ge = 64759143328$
17) $He = 705437291$
19) $Ie = 7792354$
21) $Ke = 87040$
23) $Le = 981$
25) $Me = 11$

$10\ \text{Log.}\ \frac{5}{4} = 2,2314355131420975576629507$

$A = ,0711462450592885375494071$
3) $Ae = 33345113214995643214$
5) $Be = 2813098335561819$
7) $Ce = 282525950815$
9) $De = 30896931$
11) $Ee = 3554$

$3\ \text{Log.}\ \frac{1024}{1000} = ,0711495798519481263550404$

H. Log. $10 = 2.3025850929940456840179911$ &c.

COROL. Hence therefore the Number M, mention- l before, and made Uſe of in the foregoing Examples, ill be known for the common Logarithms. For nce the common Logarithm of 10 is 1, therefore 302585 &c. $\times M = 1$. Whence $M = \frac{1}{2.302585\,\&c.}$,43429448190325182765112 89 &c.

After the ſame Manner may *Brig*'s or the common ogarithms be found, ſince we know the Number M. or let the former Series or 10 Log. $\frac{5}{4} = P$; the tter Series or Log. $\frac{2^{30}}{10^{9}} = Q$; then the Log. 2 = $\frac{\;-MP}{30}$ or $\frac{9+MQ}{30}$, whence are had the Logarithms 4 and 8; likewiſe the Log. $5 = \frac{20+MP}{30}$. And ius,

The Log. of $7 = \text{Log. } 8 \times \frac{7}{8} = \text{Log. } 8 - \frac{\frac{4}{30}M}{1}$ $\frac{\frac{4}{900}A}{3} + \frac{\frac{4}{900}B}{5} + \frac{\frac{4}{900}C}{7} + \frac{\frac{4}{900}D}{9}$ &c. by Ex- ple the 4th. And likewiſe

Log. $3 = \text{Log. } 2 \times \frac{3}{2} = \text{Log. } 2 + \frac{\frac{4}{10}M}{1} + \frac{\frac{4}{100}A}{3}$ $\frac{\frac{4}{100}B}{5} + \frac{\frac{4}{100}C}{7}$ &c. from whence is had the Log. 6 and 9.

Or the Log. $9 = \text{Log. } 10 \times \frac{9}{10} = 1 - \frac{\frac{1}{10}M}{1} +$ $A + \frac{\frac{1}{10}B}{3} + \frac{\frac{1}{10}C}{4}$ &c. and Log. 11 = Log. 10 $\frac{1}{0} = 1 + \frac{\frac{1}{10}M}{1} - \frac{\frac{1}{10}A}{2} - \frac{\frac{1}{10}B}{3} - \frac{\frac{1}{10}C}{4} -$ &c. h theſe by Example the 3d. And ſo for others.

SCHOLIUM 1.

The Rules in Example the 3d and 4th are the on
two Methods by which the Logarithm of a Numb
can be found without the Help of other given Log
rithms. And therefore to find the Log. of a gre
Number thereby, we muſt take two or more ſuc
fractional Numbers, that the Product of ſome Powe
of theſe Numbers may make the Number who
Logarithm is ſought: And then there will be ha
two or more Series, which will give the require
Logarithm; in the ſame Manner as is ſhewn in Ex.
for the Log. of the Number 10.

SCHOLIUM 2.

To find the Number from the given Logarithm.

Let n = *any Number,*
l = *Log. of it,*
$n+x$ = *any other Number,*
$l+z$ = *its Log.* = L.

$m = \frac{1}{M} =$ 2.302585 &c. *for the common Loga*

Then we ſhall have $\dot{z} = \frac{M\dot{x}}{n+x}$, or $M\dot{x} = n\dot{z} + x$
that is $\dot{x} = mn\dot{z} + mx\dot{z}$.

To find the Fluent, by Sir *I. Newton*'s Metho
(Prop. X. Rule 2.)

	$mn\dot{z}$	*	*	*
$+\ mx\dot{z}$		$+\ m^2nz\dot{z}$	$+\ \frac{m^3nz^2\dot{z}}{2}$	$+\ \frac{m^4nz^3\dot{z}}{2.3}$
$\dot{x} =$	$mn\dot{z}$	$+\ m^2nz\dot{z}$	$+\ \frac{m^3nz^2\dot{z}}{2}$	$+\ \frac{m^4nz^3\dot{z}}{2.3}$
$x =$	mnz	$+\ \frac{m^2nz^2}{2}$	$+\ \frac{m^3nz^3}{2.3}$	$+\ \frac{m^4nz^4}{2.3.4}$ &

Whi

Which is correct; becauſe when $x=0$, $z=0$.

Therefore $n+x=n\times:1+mz+\frac{\overline{mz}^2}{2}+\frac{\overline{mz}^3}{2.3}+\frac{\overline{mz}^4}{2.3.4}+\frac{\overline{mz}^5}{2.3.4.5}$ &c. that is $=n\times:1+mz+\frac{mz}{2}A+\frac{mz}{3}B+\frac{mz}{4}C+\frac{mz}{5}D$ &c. $=$ *Number whoſe Log. is* $l+z$, *where* $z=L-l$.

Cor. *If* $n=1$, *then* $l=0$, *and* $z=L$. Therefore $1+mL+\frac{mL}{2}A+\frac{mL}{3}B+\frac{mL}{4}C+\frac{mL}{5}D$ &c. $=$ *Number whoſe Log. is L.*

And if $m=1$, then

$$1+L+\frac{L^2}{2}+\frac{L^3}{2.3}+\frac{L^4}{2.3.4}+\frac{L^5}{2.3.4.5}\ \&c. =$$

Number whoſe Hyperbolic Logarithm is L.

PROB. III.

To draw Tangents to Curves.

1. In all geometrical Curves, where there is given the Relation of the Abſciſſa and Ordinate. Let $AP=x$, $PM=y$, and draw mp parallel and infinitely near to MP, and MR parallel to AP, and draw the Tangent MT, and let Pp or $MR=\dot{x}$, $Rm=\dot{y}$. Then by the ſimilar Triangles TPM and MRm, $\dot{y}:\dot{x}::y:\frac{y\dot{x}}{\dot{y}}=PT$ the Subtangent, that is $\frac{y\dot{x}}{\dot{y}}=PT$. 25.

FIG. *Therefore, by Help of the Equation of the Curve, ex-*
25. *terminate* $\dot{x}$ *or* $\dot{y}$ *out of the Quantity* $\frac{y\dot{x}}{\dot{y}}$, *and you will get the Value of the Subtangent.*

2. *In mechanical or tranſcendent Curves referred to an Axis, the Subtangent* $\frac{y\dot{x}}{\dot{y}}$ *may be clear'd of the Fluxions by the Equation of the Curve, by the Help of thoſe of the known Curve it is related to.*

26. 3. *And in any Curve QM referred to a fixt Point A; draw AT perpendicular to AM, and let* $AM = y = AR$, $Rn = y'$, $RM = x'$; *and by ſimilar Triangles* $\frac{yx'}{y'}$ or $\frac{y\dot{x}}{\dot{y}}$ $=$ *AT the Subtangent, out of which the Fluxions may be exterminated as before.*

There are ſeveral other Methods of drawing Tangents, but this is the moſt general and eaſy.

EXAMPLE I.

27. *To draw a Tangent to the Circle.* Let $AP = x$, $PM = y$, Radius $AC = r$; then will $2rx - xx = yy$, whence $2r\dot{x} - 2x\dot{x} = 2y\dot{y}$, and $\dot{x} = \frac{y\dot{y}}{r-x}$. Therefore $PT = \frac{y\dot{x}}{\dot{y}} = \frac{yy}{r-x} = \frac{2rx-xx}{r-x}$.

EX. 2.

28. *To draw a Tangent to the Ellipſis.* Let Tranſverſe $AB = 2a$, Latus rectum $= b$, $AP = x$, $PM = y$. Then by the Nature of the Curve $\frac{2ayy}{b} = 2ax - xx$, and in Fluxions $\frac{4ay\dot{y}}{b} = 2a\dot{x} - 2x\dot{x}$, or $\dot{y} = \frac{ab\dot{x} - bx\dot{x}}{2ay}$,

conſequently $PT = \frac{y\dot{x}}{\dot{y}} = \frac{2ayy}{ba-bx} = \frac{2ax-xx}{a-x}$. FIG.

Ex. 3.

25. *To draw Tangents to all ſorts of Parabolas.* Let $AP=x$, $PM=y$, and $x=y^m$, putting the Parameter $=1$. Then $\dot{x}=my^{m-1}\dot{y}$, therefore $PT=\frac{y\dot{x}}{\dot{y}} = my^m = mx$. And therefore in the common Parabola, where $m=2$, $TP=2x=2AP$.

Ex. 4.

25. *To draw a Tangent to an Hyperbola.* Let $AP=x$, $PM=y$, Tranſverſe $=2a$, Latus Rectum $=b$. Then the Property of the Figure is, $\frac{2ayy}{b} = 2ax+xx$; in Fluxions $\frac{4ay\dot{y}}{b} = 2a\dot{x} + 2x\dot{x}$ or $\dot{y} = \frac{a\dot{x}+x\dot{x}}{2ay}b$; therefore $PT = \frac{y\dot{x}}{\dot{y}} = \frac{2ayy}{ba+bx} = \frac{2ax+xx}{a+x}$.

Ex. 5.

29. *Let MB be an Hyperbola between the Aſſymptotes*; Put $AP=x$, $PM=y$, and $aa=xy$; then $x\dot{y}+y\dot{x}=0$, or $\dot{x} = \frac{-x\dot{y}}{y}$; whence $PT = \frac{y\dot{x}}{\dot{y}} = -x$, and the negative Sign ſhews that T lies on the contrary Side of PM with A.

Ex. 6.

30. *To draw a Tangent to the Ciſſoid AM.* Let $AP=x$, $PM=y$, $AB=a$, BC the Aſſymptote. Then by the Nature of the Curve $ay^2-xy^2=x^3$; whence $2ay\dot{y} - 2xy\dot{y} - y^2\dot{x} = 3x^2\dot{x}$, or $\dot{x} = \frac{2ay-2xy}{yy+3xx}\dot{y}$,

FIG. 30. whence $PT = \frac{y\dot{x}}{\dot{y}} = \frac{2ayy - 2xyy}{y^2 + 3x^2}$, or expunging yy, $PT = \frac{2ax - 2xx}{3a - 2x}$.

Ex. 7.

31. *To draw a Tangent to the Conchoid.* Let $CA = b$, $AV = a$, $AP = x$, $PM = y$. By the Nature of the Curve, $\overline{b+x}\sqrt{aa - xx} = xy$, whence $\dot{x}\sqrt{aa - xx} - \frac{x\dot{x} \times \overline{b+x}}{\sqrt{aa - xx}} = x\dot{y} + y\dot{x}$; whence $PT = \frac{y\dot{x}}{\dot{y}} = \frac{yx\sqrt{aa - xx}}{aa - 2xx - bx - y\sqrt{aa - xx}} = \frac{\overline{b+x} \times \overline{aax - x^3}}{-baa - x^3}$; and being negative, it lies on the contrary Side of PM.

Otherwiſe thus;

32. Let $AP = v$, $AM = y$, $PM = a$, $AH = b$, $MR = \acute{x}$; Then by the Nature of the Curve $v + a = y$, and $\acute{v} = \acute{y}$; and by ſimilar Triangles $Pp = \frac{v\acute{x}}{y}$, and $\acute{v} = \frac{v \times Pp}{b} = \frac{v^2\acute{x}}{by}$, and thence $\acute{x} = \frac{by\acute{y}}{vv}$ or $\dot{x} = \frac{by\dot{y}}{vv}$; therefore $AT = \frac{y\dot{x}}{\dot{y}} = \frac{byy}{vv}$.

Ex. 8.

25. *Let AM be the Catenary*, $AP = x$, $PM = y$, $AM = z$, by the Nature of the Curve $zz = 2ax + xx$, whence $z\dot{z} = a\dot{x} + x\dot{x}$, and $\dot{z} = \frac{a+x}{z}\dot{x}$; but $\dot{y}^2 = \dot{z}^2 - \dot{x}^2 = \frac{\overline{a+x}^2}{zz}\dot{x}^2 - \dot{x}^2 = \frac{\overline{a+x}^2 - zz}{zz}\dot{x}^2 =$ (expunging zz)

$zz)\ \frac{aa\dot{x}^2}{zz}$, and $\dot{y} = \frac{a\dot{x}}{z}$; therefore $PT = \frac{y\dot{x}}{\dot{y}}$ FIG. 25.
$= \frac{zy}{a}$.

Ex. 9.

To draw a Tangent to the Cycloid. Let $AP = x$, 33.
$PM = y$, $AC = 2a$, $PB = u$, Arch $AB = z$. By the Nature of the Figure $y = z + u$, and thence $\dot{y} = \dot{z} + \dot{u}$, but from the Circle $\dot{z} = \frac{a}{u}\dot{x}$, and $\dot{u} = \frac{a-x}{u}\dot{x}$; whence $PT = \left(\frac{y\dot{x}}{\dot{y}} = \frac{y\dot{x}}{\dot{z}+\dot{u}} =\right) \frac{uy}{2a-x} = \frac{AP}{PB}y$.

Ex. 10.

To draw a Tangent to the Quadratix AMB. Let 34.
$AC = b$, $CB = r$, $AP = x$, $PM = y$, $DN = z$, $NI = s$, $CM = v$, $NQ = t$, $RM = u$. Now by the Nature of the Curve $bz = ry$, therefore $b\dot{z} = r\dot{y}$, and by the Property of the Circle $r\dot{s} = t\dot{z}$; whence $\dot{z} = \frac{r\dot{y}}{b} = \frac{r\dot{s}}{t}$, and $\dot{s} = \frac{t\dot{y}}{b}$. By ſimilar Triangles $ty = su$, and thence $t\dot{y} + y\dot{t} = u\dot{s} + s\dot{u}$, that is (becauſe $u = b - x$ or $\dot{u} = -\dot{x}$, and $rr - ss = tt$ or $\dot{t} = \frac{-s\dot{s}}{t}$) $t\dot{y} - \frac{sy}{t}\dot{y} = (u\dot{s} - s\dot{x} =)\ \frac{tu\dot{y}}{b} - s\dot{x}$; whence $\dot{x} = \frac{tu\dot{y}}{bs} + \frac{y\dot{y}}{b} - \frac{t\dot{y}}{s}$. Therefore $PT = \frac{y\dot{x}}{\dot{y}} = \frac{tuy}{bs} + \frac{yy}{b} - \frac{ty}{s} =$ (becauſe $u = \frac{ty}{s}$) $\frac{u^2+y^2}{b} - u = \frac{v^2}{b} - u$.

Ex.

Ex. 11.

FIG. 35. *Let AS be any given Curve, and AM another Curve, ſuch that AP = Arch AS, and Ordinate PM = AQ, to draw the Tangent TM.*

Let $AP=x$, $PM=y$, $AQ=v$, Tangent $SR=s$, $QR=t$, $AS=z$. By the Nature of the Figure $x=z$, and $v=y$, therefore $\dot x = \dot z$ and $\dot v = \dot y$; and by ſimilar Triangles $\dot v = \frac{QR}{SR}\dot z = \frac{t\dot x}{s}$; therefore $PT = (\frac{y\dot x}{\dot y} = \frac{y\dot x}{\dot v} =) \frac{sy}{t}$.

Ex. 12.

36. *Let BM be an exponential Curve,* $AP=x$, $PM=y$, and $a^x = y$. Let A, Y be the hyperbolic Logarithms of a, y; then $xA=Y$, and $A\dot x = \dot Y = \frac{\dot y}{y}$; whence $PT = \frac{y\dot x}{\dot y} = \frac{1}{A}$ an invariable Quantity: Therefore *BM* is the Logarithmic Curve.

Ex. 13.

36. *Let the Nature of an exponential Curve be expreſſed by this Equation* $x^x = y$; Let X, Y be the hyperbolic Logarithms of x, y; then $xX = Y$, and $x\dot X + X\dot x = \dot Y$; but $\dot X = \frac{\dot x}{x}$, and $\dot Y = \frac{\dot y}{y}$, (by Prob. II.) therefore $\dot x + X\dot x = \frac{\dot y}{y}$, or $y\dot x + yX\dot x = \dot y$, whence $PT = \frac{y\dot x}{\dot y}$ $= \frac{y\dot x}{y\dot x + Xy\dot x} = \frac{1}{1+X}$.

Ex. 14.

37. *To draw a Tangent to Archimedes's Spiral.* Let Radius $AP = r$, Arch $QP = z$, $AM = y$, c a given Line.

By the Nature of the Figure $rz = cy$, and thence $\dot{z} = \frac{c\dot{y}}{r}$, and by ſimilar Triangles $\dot{x} = \frac{y\dot{z}}{r} = \frac{cy}{rr}\dot{y}$, whence $AT = \frac{y\dot{x}}{\dot{y}} = \frac{cyy}{rr}$. FIG. 37.

Ex. 15.

To draw a Tangent to the reciprocal Spiral. Let Radius $BA = a$, Arch $BC = b$, $BP = z$, $AM = y$, $Rn = \acute{x}$. The Equation of the Curve is $ab = zy$, and therefore $z\dot{y} + y\dot{z} = 0$; and by ſimilar Triangles $y\acute{z} = -a\acute{x}$, or $y\dot{z} = -a\dot{x}$, whence $z\dot{y} - a\dot{x} = 0$, or $\dot{x} = \frac{z\dot{y}}{a}$. Therefore $AT = \frac{y\dot{x}}{\dot{y}} = \frac{zy}{a} = b$, a given Quantity. 38.

Ex. 16.

Let AMD be a Spiral of ſuch a kind, that putting AM or AC $= y$, *Arch MC* $= v$, *AD* $= a$, *a given Line, and* $av^2 = y^3$. Let Arch $BE = z$, and by the ſimilar Sectors AMC, AEB, $av = yz$, whence expunging v, $z^2 = ay$, alſo rM or $\acute{x} = \frac{y\acute{z}}{a}$, or $\dot{x} = \frac{y\dot{z}}{a}$. And $2z\dot{z} = a\dot{y}$; whence $\dot{x} = \frac{y\dot{y}}{2z} = \frac{z\dot{y}}{2a}$, and therefore $AT = \left(\frac{y\dot{x}}{\dot{y}} = \frac{yz}{2a} = \frac{av}{2a} = \right) \frac{1}{2}v$. 39.

PROB.

PROB. IV.

To find the Point of Inflexion or contrary Flexure of a given Curve.

FIG. The Point of Inflexion or contrary Flexure is that Point which ſeparates the convex from the concave Part of the Curve.

40. In the Curve AM any way related to the Axis AP, let $AP = x$, Ordinate $PM = y$; draw the Ordinate pm parallel and infinitely near PM, and Mr parallel to AP. Now the Ratio of Mr to rm is the greateſt or leaſt poſſible in the Point of contrary Flexure, that is $\frac{Mr}{mr}$ or $\frac{\dot{x}}{\dot{y}}$ is a Maximum or Minimum, whence (by Prob. I.) its Fluxion $\frac{\dot{y}\ddot{x} - \dot{x}\ddot{y}}{\dot{y}^2} = 0$, or $\dot{y}\ddot{x} - \dot{x}\ddot{y} = 0$; hence if $\ddot{x} = 0$, then $\ddot{y} = 0$, or if $\ddot{y} = 0$, then $\ddot{x} = 0$.

41. In a Curve BM referred to the fixed Point P, take Mm infinitely ſmall, and draw Pm, and Mr perpendicular to it, draw the Tangent MT and PT perpendicular to it; and let $PM = y$, Curve $BM = z$, $Mr = \acute{x}$, $mr = \acute{y}$. Then, by ſimilar Triangles, $PT = \frac{y\acute{x}}{\acute{z}} = \frac{y\dot{x}}{\dot{z}}$. But in the Point of Inflexion the Perpendicular PT is a Maximum or Minimum, therefore its Fluxion or the Fluxion of $\frac{y\dot{x}}{\dot{z}} = 0$, that is $\frac{y\dot{z}\ddot{x} + \dot{x}\dot{y}\dot{z} - y\dot{x}\ddot{z}}{\dot{z}^2} = 0$, or $y\dot{z}\ddot{x} - y\dot{x}\ddot{z} + \dot{x}\dot{y}\dot{z} = 0$.

Out of which any of the Fluxions may be exterminated by Help of the Equation $\dot z^2 = \dot x^2 + \dot y^2$ and its Fluxion, making any of the Fluxions invariable, or the ſecond Fluxion $= 0$. Hence therefore to find the Point of Inflexion, the Rule is FIG.

1. *In Curves referred to an Axis, let* $AP = x$, $PM = y$, *put the Equation of the Curve into Fluxions, and again put the reſulting Equation into Fluxions, making both* $\dot x$ *and* $\dot y$ *given Quantities.* 40.

2. *In Curves related to a Pole or given Point P; let* $PM = y$, $BM = z$, *and* Mm, mr, Mr *will be as* $\dot z$, $\dot y$, $\dot x$. *Put the Equation of the Curve into Fluxions, and the reſulting Equation into Fluxions again, writing* 1 *for ſome firſt Fluxion each Operation: Then you will determine the other Fluxion which muſt be ſubſtituted into one of the following Equations that contains that other Fluxion. And then you will get the Value of* y. 41.

If $\dot x = 1$. *then* $y = \frac{1+\dot y^2}{\ddot y} = \frac{\dot z\sqrt{\dot z - 1}}{\ddot z}$.

If $\dot y = 1$. $y = \frac{\dot x + \dot x^3}{-\ddot x} = \frac{\dot z^3 - \dot z}{-\ddot z}$.

If $\dot z = 1$. $y = \frac{1 - \dot y^2}{\ddot y} = \frac{\dot x\sqrt{1 - \dot x^2}}{-\ddot x}$.

EXAMPLE I.

Let the Equation of the Curve be $ax^2 = x^2y + aay$. This in Fluxions is $2ax\dot x = 2yx\dot x + x^2\dot y + aa\dot y$; which put again into Fluxions, $2a\dot x^2 = 2y\dot x^2 + 2x\dot x\dot y + 2x\dot x\dot y = 2y\dot x^2 + 4x\dot x\dot y$, and expunging $\dot y$, $a\dot x^2 = y\dot x^2 + 2x\dot x \times \frac{2ax - 2yx}{aa + xx}\dot x$; that is by Reduction $aa + xx = 4xx$; whence $x = a\sqrt{\frac{1}{3}}$.

Ex. 2.

FIG. 42. *Let BM be Nichomedes's Concoid.* $AB = a$, $AC = b$; then its Nature is $yx = \overline{b+x}\sqrt{aa-xx}$. In Fluxions $y\dot{x} + x\dot{y} = \dot{x}\sqrt{aa-xx} - \frac{bx\dot{x} + x^2\dot{x}}{\sqrt{aa-xx}}$; and expunging y, $\frac{-x^3 - aab}{xx\sqrt{aa-xx}}\dot{x} = \dot{y}$, and this again in Fluxions is $\frac{2a^4b - a^2x^3 - 3aabxx}{\overline{aax^3 - x^5}\sqrt{aa-xx}}\dot{x}^2 = 0$, and reduced $x^3 + 3bxx = 2aab$.

Ex. 3.

43. *Let AFK be a Cycloid of ſuch a Kind, that Arch AD : DM : : ADB : BK.* Let $AB = 2r$, $ADB = a$, $BK = b$, $PD = z$, $AD = u$; then $y = z + \frac{bu}{a}$, and $\dot{y} = \dot{z} + \frac{b\dot{u}}{a}$; by the Property of the Circle $\dot{z} = \frac{r-x}{\sqrt{2rx-xx}}\dot{x}$, and $\dot{u} = \frac{r\dot{x}}{\sqrt{2rx-xx}}$, whence $\dot{y} = \frac{ar - ax + rb}{a\sqrt{2rx-xx}}\dot{x}$; in Fluxions $\frac{brx - arr - brr}{\overline{2rx-xx}^{\frac{3}{2}}}\dot{x}^2 = 0$, reduced $x = r + \frac{ar}{b}$.

Ex. 4.

44. *Let BM be the helicoid Parabola*, Radius PB or $PE = r$, Arch $BE = v$, $PM = y$. Then by the Nature of the Curve $av = \overline{r-y}^2$, whence $a\dot{v} = -2\dot{y} \times \overline{r-y}$, but by similar Sectors $\acute{v} = \frac{r\acute{x}}{y}$, or $\dot{v} = \frac{r\dot{x}}{y}$;

whence $ra\dot{x} = -2y\dot{y} \times \overline{r-y}$; and $\dot{y} = \frac{ar}{2y \times \overline{y-r}}$ (putting $\dot{x} = 1$), and $\ddot{y} = \frac{-ar\dot{y}}{2yy} \times \frac{2y-r}{\overline{y-r}^2}$, whence $y = \frac{1+\dot{y}^2}{\ddot{y}} = \frac{4yy \times \overline{y-r}^2 + aarr}{-2ar\dot{y} \times \overline{2y-r}}$, and expunging $\dot{y}$, $4y^2 \times \overline{y-r}^2 + aarr = 2ary \times \overline{2y-r} \times \frac{-ar}{2yy-2ry}$, that is $4yy \times \overline{y-r}^3 + aarr \times \overline{y-r} + aarr \times \overline{2y-r} = 0$. Which reduced gives $4y^5 - 12ry^4 + 12r^2y^3 - 4r^3y^2 + 3r^2a^2y - 2r^3a^2 = 0$.

Ex. 5.

Suppoſe BM to be a ſort of Spiral, PT or *PR* $=a$, Arch *RT* $=v$, and let $a^3 = vyy$ expreſs its Nature; then 45.
$2vy\dot{y} + yy\dot{v} = 0$, but by ſimilar Triangles $\dot{v} = \frac{a\dot{x}}{y}$, whence $2vy\dot{y} + ay\dot{x} = 0$, and $\dot{x} = \frac{-2v}{a}$ putting $\dot{y} = 1$; and expunging v, $\dot{x} = \frac{-2aa}{yy}$, and $\ddot{x} = \frac{4aa}{y^3}$; therefore $y = \frac{\dot{x}+\dot{x}^3}{-\ddot{x}} = \frac{1}{2}y + \frac{2a^4}{y^3}$, or $y^4 = 4a^4$, whence $y = a\sqrt{2}$.

PROB. V.

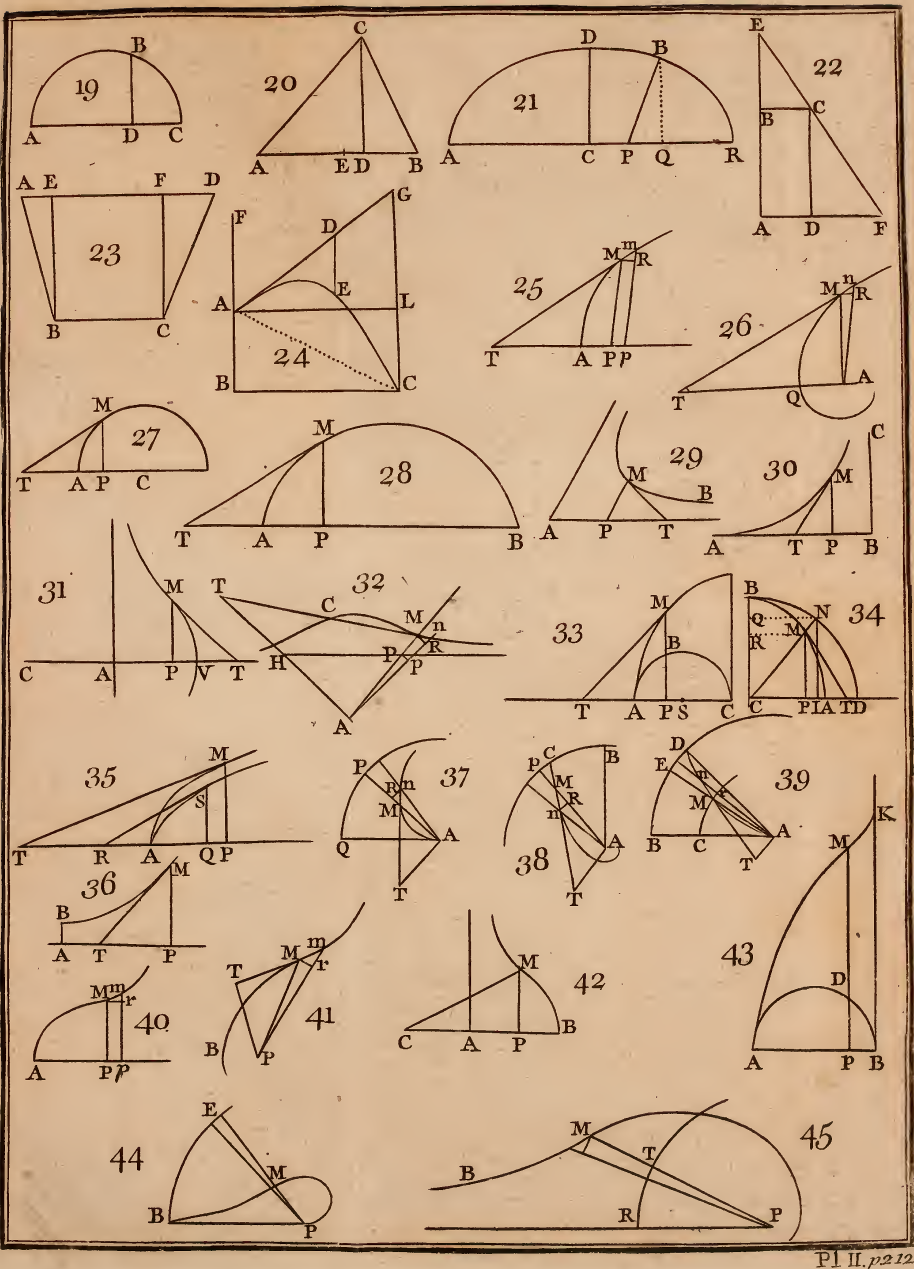
To find the Radius of Curvature in Curves.

Let *Amq* be a Curve referred to the Axis *AE*, and 46.
ſuppoſe *C* to be the Center, and *Cm* the Radius of Curvature in any Point *m*, and deſcribe the equicurve

FIG. 46. Circle *Dm* coinciding with the Curve in *mn*. Draw *DC*, *mr* parallel, and *AF* perpendicular to *AB*, and produce *mB* to *H*, and draw *nbb* parallel and infinitely near *mBH*; and call the Abſciſſa *AB*, x; perpendicular Ordinate *Bm*, y; *Am*, z; *mC*, r; *DF*, b; *AF* or *BH*, c; *mr*, $\dot{x}$; *rn*, $\dot{y}$; and *mn*, $\dot{z}$; then $CH = r - b - x$.

By the Nature of the Circle $DH \times \overline{2DC - DH} = Hm^2$, that is $2rx + 2rb - bb - 2bx - xx = cc + 2cy + yy$; which put into Fluxions $r\dot{x} - b\dot{x} - x\dot{x} = c\dot{y} + y\dot{y}$; which put into Fluxions again, $r\ddot{x} - b\ddot{x} - x\ddot{x} - \dot{x}^2 = c\ddot{y} + y\ddot{y} + \dot{y}^2$; therefore $\overline{r - b - x} \times \ddot{x} - \ddot{y} \times c + y = \dot{x}^2 + \dot{y}^2 = \dot{z}^2$. But by ſimilar Triangles $r - b - x = \frac{r\dot{y}}{\dot{z}} = \frac{r\dot{y}}{\dot{z}}$, and $c + y = \frac{r\dot{x}}{\dot{z}} = \frac{r\dot{x}}{\dot{z}}$, whence the former Equation becomes $\frac{r\dot{y}\ddot{x}}{\dot{z}} - \frac{r\dot{x}\ddot{y}}{\dot{z}} = \dot{z}^2$; which reduced gives $r = \frac{\dot{z}^3}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$ for the Radius of Curvature of the Circle *Dm* or of the Curve *Am* in the Point *m*.

Now ſince we are at Liberty to take any one of the Fluxions $\dot{x}$, $\dot{y}$, $\dot{z}$, as invariable; we can, by the Help of this Equation $\dot{z}^2 = \dot{x}^2 + \dot{y}^2$ and its Fluxion, exterminate either of the other indetermin'd Fluxions out of the Value of r, and by that Means find ſeveral different Forms for the Radius of Curvature, to ſuit different Caſes. And the ſame Way will ſeveral Values of *mH*, *CH*, be determin'd. Thus we ſhall find when $\dot{x}$ is given, the Radius $= \frac{\dot{z}^3}{-\dot{x}\ddot{y}} = \frac{\dot{z}^2\dot{y}}{-\dot{x}\ddot{z}}$; when $\dot{y}$ is given, the Radius $= \frac{\dot{z}^3}{\dot{y}\ddot{x}} = \frac{\dot{z}^2\dot{x}}{\dot{y}\ddot{z}}$; and if $\dot{z}$ be given, the Radius $= \frac{\dot{z}\dot{y}}{\ddot{x}} = \frac{\dot{z}\dot{x}}{-\ddot{y}}$.


19 A D C B
20 C A E D B
21 D B A C P Q R
22 E B C A D F
23 A E F D B C
24 F D G A E L B C
25 M m R T A P p
26 M n R T Q A
27 M T A P C
28 M T A P B
29 M B A P T
30 C M A T P B
31 M C A P V T
32 T C M n R H P p A
33 M B T A P S C
34 B Q N R M C P I A T D
35 M S T R A Q P
36 M B A T P
37 P R n M A Q T
38 p C B M R n A T
39 D E n r M B C A T
40 M m r A P p
41 T M m r B P
42 M B C A P
43 K M D A P B
44 E M B P
45 M T B R P

Pl II. p212.

Again in Spirals or Curves referr'd to a fixt Point *B*, as *AMn*. Let *C* be the Center, and *CM* the Radius of Curvature in any Point *M*; and deſcribe the Circle *DMn* coinciding with the Curve in the infinitely ſmall Part *Mn*. Draw *Bn* infinitely near *BM*, and *CD*, *Mr* perpendicular to *BM*; and call *BM*, y; Curve *AM*, z; Radius *CM*, r; *DE*, v; *Ee*, $\acute{v}$; *Mr*, $\acute{x}$; *Mn*, $\acute{z}$; *rn*, $\acute{y}$. By the ſimilar Triangles *MEC* and *Mnr*, $ME = \frac{r\acute{x}}{\acute{z}} = \frac{r\dot{x}}{\dot{z}}$, and $EC = \frac{r\acute{y}}{\acute{z}} = \frac{r\dot{y}}{\dot{z}}$, therefore $v = r - \frac{r\dot{y}}{\dot{z}}$, and thence $\dot{v} = \frac{r\dot{y}\ddot{z} - r\dot{z}\ddot{y}}{\dot{z}^2}$: And by the ſimilar Triangles *BEe*, *BMr*, it is *BE* : *BM* : : *Ee* : *Mr*, that is $y - \frac{r\dot{x}}{\dot{z}} : y :: (\acute{v} : \acute{x} :: \dot{v} : \dot{x} ::) \frac{r\dot{y}\ddot{z} - r\dot{z}\ddot{y}}{\dot{z}^2} : \dot{x}$, which multiply'd and reduced gives $r = \frac{y\dot{x}\dot{z}^2}{\dot{z}\dot{x}^2 + y\dot{y}\ddot{z} - y\dot{z}\ddot{y}}$, for the Radius of Curvature of the Circle *D M*, or of the Curve *A M* in *M*. Hence alſo $ME = \frac{y\dot{x}^2\dot{z}}{\dot{z}\dot{x}^2 + y\dot{y}\ddot{z} - y\dot{z}\ddot{y}}$, and $EC = \frac{y\dot{y}\dot{x}\dot{z}}{\dot{z}\dot{x}^2 + y\dot{y}\ddot{z} - y\dot{z}\ddot{y}}$. FIG. 47.

Now ſince we can make any one of the Fluxions $\dot{x}$, $\dot{y}$, $\dot{z}$, invariable; we can expunge either of the reſt and its Fluxion by the Help of this Equation $\dot{z}^2 = \dot{x}^2 + \dot{y}^2$ and its Fluxion, and thence obtain Variety of different Forms for each of the Quantities *CM*, *ME*, *EC*, as before.

Otherwiſe thus, when we have the Perpendicular upon the Tangent drawn to any Point *M* of the Curve, we can find the Radius of Curvature *CM* very eaſily thus; 48.

FIG. 48. thus; let *BP*, *Bp* be perpendicular to the Tangents *MP*, *np*, and let $BP = u$, $pq = \acute{u}$, the reſt as before. Then by the ſimilar Triangles *pMq*, *MCn*; and *Mrn*, *MPB*; $r\acute{u} = pM \times Mn = \acute{y}y$, or $r\dot{u} = y\dot{y}$, whence $r = \frac{y\dot{y}}{\dot{u}}$. Hence therefore to find the Radius of Curvature the Rules are,

46. 1. *In Curves whoſe Ordinates are perpendicular to the Abſciſſa, for the Curve, Abſciſſa, and Ordinate, put z, x, y: Put the Equation of the Curve (reduced as low and ſimple as poſſible) into Fluxions, writing* 1 *for any one of the firſt Fluxions contained in it; then put the reſulting Equation into Fluxions again, writing* 1 *for the ſame firſt Fluxion. From the firſt Equation the other Fluxion will be determin'd, and by the ſecond you will get the ſecond Fluxion; which being had, ſubſtitute them into one of the following Forms that contains this firſt and ſecond Fluxion.*

FORMS.

$$\text{If } \dot{x} = 1, \left\{ \begin{aligned} \text{Radius } mC &= \frac{\overline{1+\dot{y}^2}|^{\frac{3}{2}}}{-\ddot{y}} = \frac{\dot{z}^2\sqrt{\dot{z}^2-1}}{-\ddot{z}}. \\ mH &= \frac{1+\dot{y}^2}{-\ddot{y}} = \frac{\dot{z}\sqrt{\dot{z}^2-1}}{-\ddot{z}}. \\ CH &= \frac{\dot{y} \times \overline{1+\dot{y}^2}}{-\ddot{y}} = \frac{\dot{z} \times \overline{\dot{z}^2-1}}{-\ddot{z}}. \end{aligned} \right.$$

$$\text{If } \dot{y} = 1, \left\{ \begin{aligned} \text{Radius } mC &= \frac{\overline{1+\dot{x}^2}|^{\frac{3}{2}}}{\ddot{x}} = \frac{\dot{z}^2\sqrt{\dot{z}^2-1}}{\ddot{z}}. \\ mH &= \frac{\dot{x} \times \overline{1+\dot{x}^2}}{\ddot{x}} = \frac{\dot{z} \times \overline{\dot{z}^2-1}}{\ddot{z}}. \\ CH &= \frac{1+\dot{x}^2}{\ddot{x}} = \frac{\dot{z}\sqrt{\dot{z}^2-1}}{\ddot{z}}. \end{aligned} \right.$$

FIG. 46.

$$\text{If } \dot{z} = 1, \left\{ \begin{aligned} \text{Radius } mC &= \frac{\sqrt{1-\dot{x}^2}}{\ddot{x}} = \frac{\sqrt{1-\dot{y}^2}}{-\ddot{y}}. \\ mH &= \frac{\dot{x}\sqrt{1-\dot{x}^2}}{\ddot{x}} = \frac{1-\dot{y}^2}{-\ddot{y}}. \\ CH &= \frac{1-\dot{x}^2}{\ddot{x}} = \frac{\dot{y}\sqrt{1-\dot{y}^2}}{-\ddot{y}}. \end{aligned} \right.$$

But in the Vertices of Curves, where they cut the Abſciſſa at right Angles, take $\frac{y\dot{y}}{\dot{x}}$ *for the Radius of Curvature; and in the higheſt Point, where they cut the Ordinate at Right Angles, mH is the Radius.*

47. 2. *In Curves related to a fixed Point, let the Curve* $AM = z$, *Ordinate* $BM = y$, *and* $Mr = \dot{x}$. *Put the Equation of the Curve into Fluxions, and make ſome of the firſt Fluxions invariable, for which write* 1; *and the Equation being turned into Fluxions again, write* 1 *for the invariable Fluxion, as before: Thus you will determine the other Fluxion and ſecond Fluxion, whoſe Values being ſubſtituted into one of the Forms below which contains that firſt and ſecond Fluxion, you will have the Radius of Curvature.*

FORMS.

$$\text{If } \dot{x} = 1, \text{ Radius } MC = \frac{y \times \overline{1+\dot{y}^2}^{\frac{3}{2}}}{1+\dot{y}^2-y\ddot{y}} = \frac{y\dot{z}^2\sqrt{\dot{z}^2-1}}{\dot{z}\sqrt{\dot{z}^2-1}-y\ddot{z}}.$$

$$\text{If } \dot{y} = 1, \text{ Radius } MC = \frac{y \times \overline{1+\dot{x}^2}^{\frac{3}{2}}}{\dot{x}+\dot{x}^3+y\ddot{x}} = \frac{y\dot{z}^2\sqrt{\dot{z}^2-1}}{\dot{z}^3-\dot{z}+y\ddot{z}}.$$

$$\text{If } \dot{z} = 1, \text{ Radius } MC = \frac{y\sqrt{1-\dot{x}^2}}{\dot{x}\sqrt{1-\dot{x}^2}+y\ddot{x}} = \frac{y\sqrt{1-\dot{y}^2}}{1-\dot{y}^2-y\ddot{y}}.$$

$$\text{Alſo } ME = \frac{MC \times \dot{x}}{\dot{z}}, \text{ and } EC = \frac{MC \times \dot{y}}{\dot{z}}.$$

But

FIG. 48. *But when you know the Perpendicular upon the Tangent drawn to any Point of the Curve, put this Perpendicular* $= u$; *and find the Value of* $\frac{y\dot y}{u}$, *from the Equation of the Curve: And this is the Radius of Curvature.*

EXAMPLE I.

49. *Let the Equation of the Curve be* $ax - ay = x^2 + y^2$, where $AB = x$, $BM = y$, $a =$ a given Line. Let $\dot x = 1$, then putting the Equation into Fluxions $a - a\dot y = 2x + 2y\dot y$, and again $-a\ddot y = 2 + 2\dot y^2 + 2y\ddot y$; whence $\dot y = \frac{a - 2x}{a + 2y}$, and $\ddot y = \frac{-2 - 2\dot y^2}{a + 2y} = \frac{-2 \times \overline{a+2y}^2 - 2 \times \overline{a-2x}^2}{\overline{a+2y}|^3}$ $=$ (becauſe $ax - ay = x^2 + y^2$) $\frac{-4aa}{\overline{a+2y}^3}$. Whence the Radius of Curvature $= \frac{\overline{1 + \dot y^2}|^{\frac{3}{2}}}{-\ddot y} = \frac{\overline{\overline{a-2x}^2 + \overline{a+2y}^2}|^{\frac{3}{2}}}{4aa} = \frac{\overline{2a^2}|^{\frac{3}{2}}}{4aa} = \frac{a}{\sqrt{2}}$ a determinate Quantity: Therefore *Am* is the Arch of a Circle whoſe Radius is $\frac{a}{\sqrt{2}}$.

EX. 2.

50. *Let Am be an Ellipſis,* $a =$ Tranſverſe, $b =$ the Parameter, and $abx - bxx = ayy$. In Fluxions $ab - 2bx = 2ay\dot y$ (where $\dot x = 1$), and this again in Fluxions $-2b = 2a\dot y^2 + 2ay\ddot y$; hence $y = \frac{ab - 2bx}{2ay}$, and $-\ddot y = \frac{b + a\dot y^2}{ay} = \frac{4abyy + \overline{ab - 2bx}^2}{4aay^3} =$ (expunging x) $\frac{bb}{4ay^3}$.

Therefore $mH = \frac{1 + \dot y^2}{-\ddot y} = \frac{4aayy + \overline{ab - 2bx}^2}{-\ddot y \times 4aayy} =$

aabb

$\frac{aabb+4aayy-4abyy}{aabb}y = y + \frac{4a-4b}{abb}y^3$, and Radius $mC = \frac{\overline{1+\dot{y}^2}^{\frac{3}{2}}}{-\ddot{y}} = \frac{\overline{aabb+4aayy-4abyy}^{\frac{3}{2}}}{2bba^3}$. FIG. 50.

And if Am were an Hyperbola, it would be found in the ſame Manner that $mH = y + \frac{4a+4b}{abb}y^3$, and $mC = \frac{\overline{a^2b^2+4a^2y^2+4aby^2}^{\frac{3}{2}}}{2bba^3}$.

Or thus in particular Numbers.

Let $a=3$, $b=1$, and aſſume $x=1$, $\dot{x}=1$, then $y = \sqrt{bx - \frac{b}{a}xx} = \sqrt{\frac{2}{3}}$, and by the foregoing operation $\dot{y} = \frac{ab-2bx}{2ay} = \frac{1}{6\sqrt{\frac{2}{3}}}$, and $-\ddot{y} = \frac{b+a\dot{y}^2}{ay} = \frac{3\sqrt{3}}{8\sqrt{2}}$. Whence $mC = \frac{\overline{1+\dot{y}^2}^{\frac{3}{2}}}{-\ddot{y}} = \frac{125}{54}$.

Otherwiſe thus:

Let Tranſverſe $=2r$, Conjugate $=2c$, B the Focus, $BM=y$, $BT=u$ the Perpendicular on the Tangent at M. By the Conic Sections $u = \frac{cy}{\sqrt{2ry-yy}}$, and $\dot{u} = \frac{cr\dot{y}}{\overline{2ry-yy}^{\frac{3}{2}}}$ whence $-\frac{y\dot{y}}{\dot{u}} = \frac{\overline{2ry-yy}^{\frac{3}{2}}}{cr}$, the Radius of Curvature in M. 57.

Ex. 3.

In the Parabola $ax=yy$, then $a\dot{x}=2y$, and $a\ddot{x}=2$ (putting $\dot{y}=1$); whence Radius $mC = \frac{\overline{1+\dot{x}^2}^{\frac{3}{2}}}{\ddot{x}} = \frac{\overline{1+\frac{4yy}{aa}}^{\frac{3}{2}}}{\frac{2}{a}} = \frac{\overline{aa+4yy}^{\frac{3}{2}}}{2aa} = \frac{4\times\overline{mD}^3}{aa}$. 52.

Otherwiſe thus:

FIG. 58. Let a = Latus rectum, B the Focus, $BM=y$, $BT=u$, the Perpendicular on the Tangent at M. By the Nature of the Figure $uu=\frac{1}{4}ay$, and $\dot{u}=\frac{a\dot{y}}{4\sqrt{ay}}$, whence $\frac{y\dot{y}}{\dot{u}}=\frac{4y\dot{y}\sqrt{ay}}{a\dot{y}}=\frac{4y\sqrt{ay}}{a}=\frac{8uy}{a}$, the Radius of Curvature in M.

Ex. 4.

51. *Let Dm be an Hyperbola between the Aſſymptotes.* $AB=x$, $Bm=y$. Draw mR parallel to the other Aſſymptote; then ſince the Poſition of the Ordinate mR is given, there is given the Ratio of mR and RB to mB perpendicular to AB: Let $RB=sy$, $mR=ty$, $AR=x-sy$. By the Nature of the Figure $AR\times Rm$ $=aa=txy-tsyy$, which turned into Fluxions $tx+ty\dot{x}-2tsy=0$, and again $2t\dot{x}+ty\ddot{x}-2ts=0$; hence $\dot{x}=\frac{2sy-x}{y}$, and $\ddot{x}=\frac{2s-2\dot{x}}{y}=\frac{2x-2sy}{yy}$. Then $\frac{\overline{1+\dot{x}^{2}}|^{\frac{3}{2}}}{\ddot{x}}=\frac{\overline{yy+\overline{2sy-x}|^{2}}|^{\frac{3}{2}}}{2xy-2syy}$, the Radius of Curvature in m.

And in a right-angled Hyperbola, $s=0$, and the Radius becomes $\frac{\overline{yy+xx}|^{\frac{3}{2}}}{2xy}=\frac{\overline{yy+xx}|^{\frac{3}{2}}}{2aa}=\frac{Am^{3}}{2aa}$.

Ex. 5.

52. *Let Am be the Ciſſoid,* whoſe Equation is $axx-yx^{2}$ $=y^{3}$, then $a-y=\frac{y^{3}}{xx}$, in Fluxions $-1=\frac{3yy}{xx}-\frac{2y^{3}\dot{x}}{x^{3}}$, reduced $-x^{3}=3y^{2}x-2y^{3}\dot{x}$, in Fluxions $-3x^{2}\dot{x}=6yx+$ $3y^{2}\dot{x}-6y^{2}\dot{x}-2y^{3}\ddot{x}$; hence $\dot{x}=\frac{3xy^{2}+x^{3}}{2y^{3}}$, and $\ddot{x}$

=

$= \frac{6xy+\overline{3xx-3yy}\times\dot{x}}{2y^3} = \frac{3xy^4+6x^3y^2+3x^5}{4y^6}$. Then FIG. 52. the Radius $= \frac{\overline{1+\dot{x}^2}|^{\frac{3}{2}}}{\ddot{x}} = \frac{\overline{4y\ +9x^2y^4+6x^4y^2+x^6}|^{\frac{3}{2}}}{2y^3\times:3xy^4+6x^3y^2+3x^5}$

$=$ (expunging y^3, y^6) $\frac{\overline{4aa-8ay+10yy+xx+9ay-9yy}|^{\frac{3}{2}}\times x}{6aay}$

$= \frac{\overline{4aa+ay+y^2+x^2}|^{3}\times x}{6aay} =$ (expunging xx) $\frac{\overline{4a-3y}|^{\frac{3}{2}}\times a\sqrt{y}}{6\times\overline{a-y}|^{2}}$. Therefore in the Vertex *A*, the Radius of Curvature is 0.

Or thus definitively in Numbers.

Let $a=10$, $\dot{y}=1$, and aſſume $y=2$, then $x=1$, and $\dot{x} = \frac{3xy^2+x^3}{2y^3} = \frac{13}{16}$. And $\ddot{x} = \frac{6xy+\overline{3x^2-3yy}\times\dot{x}}{2y^3} = \frac{75}{16\times16}$; whence $\frac{\overline{1+\dot{x}^2}|^{\frac{3}{2}}}{\ddot{x}} = \frac{85}{48}\sqrt{17}$.

Ex. 6.

Let $x=y^n$ *be an Equation for all Parabolas*, then (if $\dot{y}=1$) $\dot{x}=ny^{n-1}$, $\ddot{x}=\overline{nn-n}\times y^{n-2}$. Therefore $\frac{\overline{1+\dot{x}^2}|^{\frac{3}{2}}}{\ddot{x}} = \frac{\overline{1+nny^{2n-2}}|^{\frac{3}{2}}}{n.\overline{n-1}.y^{n-2}}$.

Hence it will eaſily appear that every Parabola, except the Appolonian, has the Radius of Curvature in the Vertex either infinite or nothing at all.

Cor. If $ax=yy$, $bx=y^3$, $cx=y^4$, $dx=y^5$, &c. be a Series of Parabolas, then the Angle of Contact (made with the Tangent and Curve) at the Vertex of any one is infinitely greater than the Angle of Contact of the next following one. The Angle of Contact of the firſt $ax=y^2$, is of the ſame kind with that of Circles.

Now

FIG. Now it appears that the Angles of Contact of one kind infinitely exceed thoſe of another kind, ſince the Radius of Curvature of one kind is infinitely greater than that of another; and the Curvature or Angle of Contact is reciprocally as that Radius. Thus in the Curve $cx = y^4$, the Radius $= \frac{c}{12yy} =$ Infinity, and in the Curve $dx = y^5$, the Radius $= \frac{d}{20y^3} =$ Infinity, becauſe $y = 0$; and the latter Radius is infinitely greater than the former, for $\frac{d}{20y^3} = \frac{c}{12yy} \times \frac{12d}{20cy}$; and therefore the Angle of Contact of the firſt infinitely exceeds the Angle of Contact of the laſt. Therefore a Curve of one kind, how great ſoever it may be, cannot be interpoſed at the Point of Contact between the Curve of another kind and its Tangent, however ſmall that Curve may be: Or an Angle of Contact of one kind cannot neceſſarily contain an Angle of Contact of another kind, as a whole contains a part.

Likewiſe in this Series of Parabolic Curves $ax = y^{\frac{1}{2}}$, $bx = y^{\frac{2}{3}}$, $cx = y^{\frac{3}{4}}$, $dx = y^{\frac{4}{5}}$, *&c.* the Angle of Contact which any Curve makes with the Abſciſſa at the Vertex is infinitely greater than the next preceding. The Angle of Contact of the firſt $ax = y^{\frac{1}{2}}$ is of the ſame kind with Circles. And though the Angles of Contact of the ſucceeding Curves do infinitely exceed the preceding ones, yet they can never arrive at the Magnitude of right lined Angles. Moreover between the Angles of Contact of any two of theſe may other Angles of Contact be found, *ad infinitum*, that will infinitely exceed each other.

Ex. 7.

54. *Let Dm be the Logarithmic Curve, whoſe Equa*[illegible] *is* $y\dot{x} = a\dot{y}$, putting $AB = x$, $Bm = y$, $a =$ Subtangent.

Then (if $\dot{x}=1$) $\dot{y}=\frac{y}{a}$, and $\ddot{y}=\frac{\dot{y}}{a}=\frac{y}{aa}$; FIG. 54. Therefore $\frac{\overline{1+\dot{y}^{2}}\rvert^{\frac{3}{2}}}{-\ddot{y}}=\frac{\overline{aa+yy}\rvert^{\frac{3}{2}}}{-ay}$, the Radius of Curvature; where the negative Sign only ſhews its Poſition.

Ex. 8.

Let Am be the Catenary, AB$=x$, *Bm*$=y$, *Am*$=z$; 52. Its Equation is $zz=2ax+xx$, then (putting $\dot{z}=1$) its Fluxion is $z=a\dot{x}+x\dot{x}$, and again $1=a\ddot{x}+x\ddot{x}+\dot{x}^{2}$; hence $\dot{x}=\frac{z}{a+x}$, and $\ddot{x}=\frac{1-\dot{x}^{2}}{a+x}=\frac{\overline{a+x}^{2}-zz}{\overline{a+x}^{3}}=\frac{aa}{\overline{a+x}^{3}}$. Therefore $\frac{\sqrt{1-\dot{x}^{2}}}{\ddot{x}}=\frac{\overline{a+x}^{3}\sqrt{\overline{a+x}^{2}-zz}}{\overline{a+x}\times aa}=\frac{\overline{a+x}^{2}}{aa}\sqrt{aa}=\frac{\overline{a+x}^{2}}{a}=a+\frac{zz}{a}$.

Ex. 9.

Let Am be the Cycloid, AB$=x$, *Bm*$=y$, Arch *AD* $=v$, *AR*$=a$, then *BD* $=\sqrt{ax-xx}$, and $\dot{v}=\frac{a}{2\sqrt{ax-xx}}$, by the Nature of the Circle (putting $\dot{x}=1$). By the Property of the Cycloid, $y=v+\sqrt{ax-xx}$, whence $\dot{y}=\dot{v}+\frac{a-2x}{2\sqrt{ax-xx}}=\frac{a-x}{\sqrt{ax-xx}}=\sqrt{\frac{a-x}{x}}$; and $\ddot{y}=\frac{-a}{2x^{\frac{3}{2}}\sqrt{a-x}}$; whence the Radius $=\frac{\overline{1+\dot{y}^{2}}\rvert^{\frac{3}{2}}}{-\ddot{y}}=2\sqrt{aa-ax}=2DR$. 55.

Ex. 10.

Ex. 10.

FIG. 53. *Let Dm be the Conchoid, RA* $=b$, $AD=c$, $AB=x$, $Bm=y$; its Equation is $\overline{b+y}\sqrt{cc-yy}=xy$. This ſquared and divided by yy, $\frac{bbcc}{yy}+\frac{2bcc}{y}+cc-bb-2by-yy=xx$. In Fluxions, $-\frac{2bbcc}{y^3}-\frac{2bcc}{yy}-2b-2y=2x\dot{x}$; and again $\frac{6bbcc}{y^4}+\frac{4bcc}{y^3}-2=2\dot{x}^2+2x\ddot{x}$; whence $\dot{x}=\frac{-bbcc}{y^3x}-\frac{bcc}{yyx}-\frac{b}{x}-\frac{y}{x}$, and $\ddot{x}=\frac{3bbcc}{y^4x}+\frac{2bcc}{y^3x}-\frac{1}{x}-\frac{\dot{x}^2}{x}$. Now to find the Radius of Curvature in any given Point m; let $b=5$, $c=10$, if you take $y=6$, then $x=14\frac{2}{3}$, and $\dot{x}=-2\frac{35}{72}$, and $\ddot{x}=\frac{499}{1056}$; whence $\frac{\overline{1+\dot{x}^2}^{\frac{3}{2}}}{\ddot{x}}=40{,}326$ the Radius of Curvature in that Point m.

The Radius of Curvature may alſo be expreſſed indefinitely in the ſame Manner as in the foregoing, tho' more prolixly. Let $\dot{x}=\frac{p}{x}$, $\ddot{x}=\frac{q}{x}-\frac{\dot{x}^2}{x}=\frac{qxx-pp}{x^3}$, then $\overline{1+\dot{x}^2}^{\frac{3}{2}}=\frac{\overline{xx+pp}^{\frac{3}{2}}}{x^3}$: Whence $\frac{\overline{1+\dot{x}^2}^{\frac{3}{2}}}{\ddot{x}}=\frac{\overline{xx+pp}^{\frac{3}{2}}}{qxx-pp}$; therefore either x or y being given, the other will be known, and thence the Quantities p, q. For Example, in the Vertex where $x=0$, the Radius becomes $\frac{p^3}{-pp}=-p=$

bbcc

$\frac{bbcc}{y^3} + \frac{bcc}{yy} + b + y = \frac{bb}{c} + 2b + c =$ FIG.
$\frac{\overline{b+c}^2}{c}$.

Ex. 11.

Let Dm be the Quadratrix, AB=x, Bm=y, AF or 56.
AR$=r$, *AE*$=u$, *CE*$=v$, Arch *CF*$=t$, *RCF*$=q$, *RC*$=q-t$, *Am*$=s$. By the Nature of the Circle, the Fluxion of *RC* $(-\dot{t})$ is to the Fluxion of *AE* $(\dot{u})$ as r to v, that is, $-v\dot{t} = r\dot{u}$, likewiſe $r\dot{v} = u\dot{t}$, and by the Property of the Figure $t=y$, and and $\dot{t}=\dot{y}$; and by ſimilar Triangles $uy=vx$, in Fluxions $y\dot{u} + u\dot{y} = x\dot{v} + v\dot{x}$, and expunging $\dot{v}$, $\dot{u}$, v, we get $-y^2\dot{y} + rx\dot{y} - x^2\dot{y} = ry\dot{x} = -y^2 + rx - xx$ (putting $\dot{y} = 1$), this in Fluxions $ry\ddot{x} + r\dot{x} = -2y + r\dot{x} - 2x\dot{x}$. Therefore $\dot{x} = \frac{-y^2+rx-xx}{ry} = \frac{rx-ss}{ry}$; and $\ddot{x} = \frac{-2y-2x\dot{x}}{ry} = \frac{2ssx-2ssr}{rryy}$;

hence $\frac{\overline{1+\dot{x}^2}|^{\frac{3}{2}}}{\ddot{x}} = \frac{\overline{rryy+\overline{rx-ss}^2}|^{\frac{3}{2}} \times rryy}{r^3y^3 \times \overline{2ssx-2ssr}} = \frac{-s}{2ry}$
$\times \frac{\overline{rr-2rx+ss}|^{\frac{3}{2}}}{r-x}$.

If the Radius of Curvature is required in the Vertex *F*, it will be found by expreſſing *AE* and *CE* by infinite Series in Terms of y, rejecting the ſuperfluous Terms. Thus it will hereafter be ſhewn (Ex. 5. Prob. VIII.) that $CF = CE + \frac{CE^3}{6rr}$ &c. hat is $y = v + \frac{v^3}{6rr}$, whence by Reverſion of

Series

FIG. 56. Series $v = y - \frac{y^3}{6rr}$, and thence $u = \sqrt{rr - vv} = r - \frac{yy}{2r}$, whence $uy = vx$ becomes $ry - \frac{y^3}{2r} = yx - \frac{y^3}{6rr}$, and $x = r - \frac{yy}{3r}$, and $\dot{x} = - \frac{2y\dot{y}}{3r}$. But the Radius of Curvature in F is $\frac{y\dot{y}}{\dot{x}}$, where you muſt write $r - x$ for x, $-\dot{x}$ for $\dot{x}$. Then $\frac{y\dot{y}}{\dot{x}} = \frac{y\dot{y}}{2y\dot{y}} \times 3r = \frac{3}{2}r$.

Ex. 12.

59. *Let AM be the Logarithmetic Spiral*; *TM* a Tangent; *BT*, *BP* perpendicular to *BM TM*. Let $BM = y$, $BP = u$, the given Ratio of MT to BT as c to t; then $u = \frac{ty}{c}$, and $\dot{u} = \frac{t\dot{y}}{c}$; whence $MC = \frac{y\dot{y}}{\dot{u}} = \frac{cy}{t}$; therefore C falls in the Line TB produced.

Ex. 13.

60. *Let BM be Archimedes' Spiral*, $AB = a$, Arch $AD = v$, $BM = y$, and let $by = av$, whence $b\dot{y} = a\dot{v}$; but by ſimilar Sectors $\acute{v} = \frac{a\acute{x}}{y}$ or $\dot{v} = \frac{a\dot{x}}{y}$, therefore $aa\dot{x} = by\dot{y}$. Let $\dot{y} = 1$, then $\dot{x} = \frac{by}{aa}$, and $\ddot{x} = \frac{b\dot{y}}{aa}$; whence the Radius $= \frac{y \times \overline{1 + \dot{x}^2}^{\frac{3}{2}}}{x + x^3 + y\ddot{x}} =$

FIG.

$$= \frac{\overline{a^4 + bbyy}\,^{\frac{3}{2}}}{b^3y^2 + 2ba^4} = \frac{a \times \overline{a^2 + vv}\,^{\frac{3}{2}}}{bv^2 + 2ba^2}.$$

Ex. 14.

Suppoſe the Equation of the Spiral to be $by^m = a^m v$, 60.

then, if $\dot{x} = 1$, $mby^{m-1}\dot{y} = a^m \dot{v} = \frac{a^{m+1}\dot{x}}{y}$, or $mby^m\dot{y} = a^{m+1}$; again, in Fluxions, $m^2by^{m-1}\dot{y}^2 + mby^m\ddot{y} = 0$; hence $\dot{y} = \frac{a^{m+1}}{mby^m}$, and $\ddot{y} = \frac{-a^{2m+2}}{mb^2y^{2m+1}}$. Whence the Radius $= \frac{y \times \overline{1 + \dot{y}^2}\,^{\frac{3}{2}}}{1 + \dot{y}^2 - y\ddot{y}} =$

$$\frac{\overline{m^2b^2y^{2m} + a^{2m+2}}\,^{\frac{3}{2}}}{m^3b^3y^{3m-1} + \overline{m+1} \times mby^{m-1}a^{2m+2}}.$$

Ex. 15.

Let BM be the reciprocal or hyperbolic Spiral, Radius 61. *AB* or *BD* $= a$, given Arch $AE = n$, Arch $AF = v$, $BM = y$. By the Property of the Curve $an = vy$, whence $y\dot{v} + v\dot{y} = 0$, and $\dot{v} = \frac{-v\dot{y}}{y} = \frac{-an\dot{y}}{yy}$: But, by ſimilar Triangles, $\dot{v} = \frac{a\dot{x}}{y}$, therefore $y\dot{x} = -n\dot{y}$, let $\dot{x} = 1$, then $\dot{y} = \frac{-y}{n}$, and $\ddot{y} = \frac{-\dot{y}}{n} = \frac{y}{nn}$. Therefore $\frac{y \times \overline{1 + \dot{y}^2}\,^{\frac{3}{2}}}{1 + \dot{y}^2 - y\ddot{y}} = \frac{y \times \overline{n^2 + y^2}\,^{\frac{3}{2}}}{n^3}$ $=$ (becauſe the Subtangent $BT = n$) $\frac{y \times TM^3}{n^3} = \frac{y \times GM^3}{y^3} = \frac{GM^3}{y^2}$, ſuppoſing *BG* perpendicular to *BM*.

Ex. 16.

FIG. 62. *Let BM be a kind of Spiral, Radius BD* $=a$, *Arch DF* $=t$, *BM* $=y$, *Arch MR* $=v$, *and let its Nature b* $a^2y=v^3$. By the ſimilar Sectors BMR and BFD $v=\frac{ty}{a}$, whence $a^5=t^3y^2$, in Fluxions $3y^2t^2\dot{t}+$ $2t^3y\dot{y}=0$, by ſimilar fluxionary Triangles $\dot{t}=\frac{a\dot{x}}{y}$ and expunging $\dot{t}$, $3a\dot{x}+2t\dot{y}=0$, let $\dot{y}=1$, then $\dot{x}=$ $\frac{-2t}{3a}$, and $\ddot{x}=\frac{-2\dot{t}}{3a}=\frac{-2\dot{x}}{3y}=\frac{4t}{9ay}=$ $\frac{4v}{9yy}$: Therefore $\frac{y\times\overline{1+\dot{x}^2}^{\frac{3}{2}}}{\dot{x}+\dot{x}^3+y\ddot{x}}=\frac{y\times\overline{9yy+4vv}}{-v\times\overline{6yy+8v}}$

$$=\frac{-v^3}{a^4}\times\frac{\overline{9v^4+4a^4}^{\frac{3}{2}}}{6v^4+8a^4}.$$

Or thus, in particular Numbers:

Let $y=1$, $v=4$, $\dot{y}=1$, then $\dot{x}=\frac{-2t}{3a}=$ $\frac{-2v}{3y}=\frac{-8}{3}$, and $\ddot{x}=\frac{-2\dot{x}}{3y}=\frac{16}{9}$. Whenc $\frac{y\times\overline{1+\dot{x}^2}^{\frac{3}{2}}}{\dot{x}+\dot{x}\;+y\ddot{x}}=\frac{-73\sqrt{73}}{536}$ the Radius of Curvature, where the negative Sign relates only to th Poſition of it.

SCHOLIUM.

63. Let there be any geometrical Curve defined by thi general Equation $e+fx^m+gy+bx^ry^s=0$; make A $=x$, $BM=y$, the Perpendicular to the Curve $MQ=\pi$ Then the Radius of Curvature may be expreſſed i algebraic Terms affected with π, thus; compute th Valu

FIG. 63.

Value of $\ddot{y}$ in the foregoing general Equation (putting $\dot{x}$ invariable), which ſubſtitute in the Quantity $\frac{\dot{z}^3}{-\dot{x}\ddot{y}}$, and put $\frac{\pi}{y}$ for $\frac{\dot{z}}{\dot{x}}$; and the Radius of Curvature will be =

$$\frac{\overline{ngy^{n-1}+shx^ry^{s-1}}^3\times\pi^3}{\left.\begin{array}{l}=\overline{m.\overline{m-1}.fx^{m-2}+r.\overline{r-1}.hx^{r-2}y^s}\times\overline{ngy^{n-1}+shx^ry^{s-1}}^2\\-2rshx^{r-1}y^{s-1}\times\overline{mfx^{m-1}+rhx^{r-1}y^s}\times\overline{ngy^{n-1}+shx^ry^{s-1}}\\+\overline{n.\overline{n-1}.gy^{n-2}+s.\overline{s-1}.hx^ry^{s-2}}\times\overline{mfx^{m-1}+rhx^{r-1}y^s}^2\end{array}\right\}\times y^3}$$

EXAMPLE.

Let $bx^2-ay^2+abx=0$, *repreſent an Hyperbola*, a = *Tranſverſe*, b = *Parameter*. Comparing this with the general Form, we ſhall find $e=0$, $f=b$, $m=2$, $=g$ $-a$, $n=2$, $h=ab$, $r=1$, $s=0$, whence the Radius of Curvature $=\frac{\overline{-2ay}^3\times\pi^3}{2b\times\overline{-2ay}^2-2a\times\overline{2bx+ab}^2}\times\frac{1}{:\times y^3}$ $=$ (expunging x) $\frac{\overline{-2ay}^3\,\pi^3}{-2a^3b^2y^3}=\frac{4\pi^3}{bb}$. And it will be the ſame for the Ellipſis.

PROB. VI.

To determine the (Quality or Degree of) Variation of Curvature in any Point of a Curve.

The Variation of Curvature depends on the Moment of the Radius of Curvature; and the Moment of

FIG. of the Curve; and it is as the Moment of the Radius of Curvature if the Moment of the Curve is given; and reciprocally as the Moment of the Curve if the Moment of the Radius is given. Therefore this Variation is as the Fluxion of the Radius of Curvature divided by the Fluxion of the Curve: Therefore,

1. *Find the Radius of Curvature by the laſt Problem, which divide by the Fluxion of the Curve; and the Fluxions may be exterminated by the Equation of the given Curve, or perhaps by expreſſing their Ratio by Help of the Tangent, Ordinate, or Subnormal.*

2. In Curves referred to an Axis, finding the Fluxion of the Quantity $\frac{\dot{z}^3}{-\ddot{y}}$, which divide by $\dot{z}$, putting $\dot{x}=1$, and exterminating $\dot{z}$, $\ddot{z}$, you will get the following Rule. *Put x for the Abſciſſa, y for the Ordinate, $\dot{x}=1$: Then ſubſtitute the Values of $\dot{y}$, $\ddot{y}$, $\dddot{y}$ (got from the Equation of the Curve) into the Quantity* $\frac{-3\dot{y}\ddot{y}^2+\dddot{y}\times\overline{1+\dot{y}^2}}{\ddot{y}^2}$, *and it will give the Variation.*

EXAMPLE I.

52. *Let the Curve be a Parabola,* then $ax=yy$, and $\dot{y}=\frac{a}{2y}$ $\ddot{y}=\frac{-a\dot{y}}{2yy}$, $\dddot{y}=\frac{2a\dot{y}^2-ay\ddot{y}}{2y^3}$. Now if $a=2$, $x=\frac{1}{2}$, $y=1$, then $\dot{y}=1$, $\ddot{y}=-1$, $\dddot{y}=3$, whence $\frac{-3\dot{y}\ddot{y}^2+\dddot{y}\times\overline{1+\dot{y}^2}}{\ddot{y}^2}=3$. Likewiſe if $x=2$, $y=2$, then the Variation $=6$.

In general,

Since $ax=y^2$, therefore $\dot{y}=\frac{a}{2y}$, and $\ddot{y}=\frac{-a\dot{y}}{2yy}=\frac{-aa}{4y^3}$, and $\dddot{y}=\frac{2a\dot{y}^2-ay\ddot{y}}{2y^3}=\frac{3aa\dot{y}}{4y^4}=\frac{3a^3}{8y^5}$.

Whence

FIG. 52.

Whence $\frac{-3\dot{y}\ddot{y}^2+\dddot{y}\times\overline{1+\dot{y}^2}}{\ddot{y}^2}=\frac{-3a^5}{32y^7}+\frac{3a^3}{8y^5}+\frac{3a^5}{32y^7}:\times\frac{16y^6}{a^4}=\frac{6y}{a}$.

Ex. 2.

50. *In the Ellipſis let* $bx-\frac{b}{a}xx=yy$. Suppoſe $a=\frac{2}{3}$, $b=2$, to find the Variation of Curvature when $x=\frac{1}{2}$, and $y=\frac{1}{2}$. By the Proceſs of Example 2. of the laſt Problem, $\dot{y}=\frac{ab-2bx}{2ay}=-1$, $\ddot{y}=\frac{b+a\dot{y}^2}{-ay}=-8$, and it will be found that $\dddot{y}=\frac{b+a\dot{y}^2-2ay\ddot{y}}{ayy}\times\dot{y}=-48$. Whence $\frac{-3\dot{y}\ddot{y}^2+\dddot{y}\times\overline{1+\dot{y}^2}}{\ddot{y}^2}=\frac{3}{2}$, the Variation required.

Or in general thus:

By the laſt Problem the Radius of Curvature is $\frac{\overline{aabb+4aayy-4abyy}|^{\frac{3}{2}}}{2bba^3}$, and its Fluxion divided by $\dot{z}$ is $\frac{a-b}{bbaa\dot{z}}\times 6y\dot{y}\sqrt{aabb+4aayy-4abyy}$. But from the Equation of the Curve $\frac{\dot{y}}{\dot{z}=\sqrt{\dot{x}^2+\dot{y}^2}}=\frac{ab-2bx}{\sqrt{aabb+4aayy-4abyy}}$; whence the Variation of Curvature becomes $\frac{a-b}{baa}\times\overline{\tfrac{1}{2}a-x}\times 12y$; and is therefore every where as the Rectangle *mBO*, ſuppoſing *O* the Center.

Ex. 3.

Ex. 3.

FIG. 52. *Let* $x = y^3$ *denote a cubic Parabola*; then $3y^2\dot{y} = 1$, and $\dot{y} = \frac{1}{3yy}$, $\ddot{y} = \frac{-2\dot{y}}{3y^3} = \frac{-2}{9y^5}$, $\dddot{y} = \frac{10\dot{y}}{9y^6} = \frac{10}{27y^8}$; whence $-3\dot{y} + \frac{\dddot{y}}{\ddot{y}^2} \times \overline{1+\dot{y}^2} = \frac{15y^2}{2} - \frac{1}{6yy}$ the Variation required.

Ex. 4.

55. *In the Cycloid Am* the Radius of Curvature is $2\sqrt{aa-ax}$, and its Fluxion $= \frac{-a\dot{x}}{\sqrt{aa-ax}}$; but by the Property of the Figure $\dot{z} = \frac{AD\times\dot{x}}{x} = \frac{DR\times\dot{x}}{y} = \frac{\dot{x}\sqrt{aa-ax}}{y}$, whence $\frac{-a\dot{x}}{\dot{z}\sqrt{aa-ax}} = \frac{-ay}{aa-ax} = \frac{-y}{BR} = -\frac{AB}{BD}$. The negative Sign only ſhews that the Curvature increaſes.

Ex. 5.

59. *In the Logarithmic Spiral*, (by Ex. 12. Prob. V.) the Radius of Curvature is $\frac{cy}{t}$ (putting $c : t : d :: TM : TB : BM$), and its Fluxion is $\frac{c\dot{y}}{t}$, and $\dot{z} = \frac{c\dot{y}}{d}$; therefore $\frac{c\dot{y}}{t\dot{z}} = \frac{d}{t}$ for the Variation of Curvature, which therefore is uniform.

Ex. 6.

61. *Let BM be the hyperbolic Spiral*, (Ex. 15. Prob. laſt) the Radius of Curvature is $\frac{y\times\overline{n^2+y^2}^{\frac{3}{2}}}{n^3}$, whoſe

Fluxion is $\frac{n^2+4y^2}{n^3}\dot{y}\sqrt{n^2+y^2}$, but the Subtangent FIG. 61. $BT=n$, therefore $\dot{z}=\frac{\dot{y}}{y}\sqrt{nn+yy}$, by which divide the other Quantity, then $\frac{n^2y+4y^3}{n^3}=$ Variation of Curvature in *M*.

PROB. VII.

To find the Nature of the Curve, by whose Evolution a given Curve is deſcribed.

A Curve *ADM* is ſaid to be deſcribed by the Evolution of another Curve *VEC*, when a right Line is ſuppoſed to be apply'd to the convex Side of the Curve *VEC*, and the End of it *A* is made to move with a circular Motion, and to deſcribe the Curve *ADM*; whilſt the Part *DE* or *MC* that continually leaves the Curve *VEC* is extended into a right Line. The Curve *VEC* is called the Evolute of *ADM*. 64.

Hence it appears that the Line *CM* which is a Tangent at *C* to the Curve *VEC*, is the Radius of Curvature in *M* of the Curve *ADM*, and is equal in Length to the Curve *CE* + the right Line *ED* the Tangent of *E*, or to the Length of the Curve *CEV* + the Line *VA*. Now the Evolute *VEC* being the Locus of the Center of Curvature of all the Points of the Curve *ADM*, the Nature of the Curve *VEC* is to be found. Therefore

1. *In Curves related to an Axis, put the Abſciſſa* $AB=x$, *perpendicular Ordinate* $BM=y$, *and let CM be the Radius of Curvature, and draw CH parallel to AB,*

FIG. *AB, and CL to BM, and produce MB to H, and put v, u for the Abſciſa and Ordinate of the Curve VC refered to the Axis AL: Then find MH, HC by Prob.* V. *and expunging x and y, the Nature of the Curve VC will be diſcovered.*

47. 2. *And in Spirals find the Radius of Curvature MC by Prob,* V. *which will give all the Points C in the Evolute.*

Thus we ſhall have $EC = \frac{MC\times\dot{y}}{\dot{z}}$, $ME = \frac{MC\times\dot{x}}{\dot{z}}$, *and therefore* $BE = y - \frac{MC\times\dot{x}}{\dot{z}}$, *and then* $BC = \sqrt{yy + MC^2 - \frac{2y\dot{x}}{\dot{z}}\times MC}$.

COROL. I. Hence it will be eaſy to find any Number of Curves whoſe Lengths may be expreſs'd by finite Equations; by aſſuming any Curve *AM* at Pleaſure and finding its Evolute *EC*: For the Radius of Curvature *MC* will give the Length of the Evolute; or $MC - DE =$ Length of the Part *EC* of the Curve.

64. COR. 2. Hence alſo if the Curve *VEC* be given, the Curve *AM* deſcribed by its Evolution may be found.

For let $AV = b$, $VEC = s$, then $AV + VC = CM = b + s$, and by ſimilar Triangles $\dot{s} : \dot{u} :: b+s : MH = \frac{b+s}{\dot{s}}\dot{u}$, therefore *MB* or $y = \frac{b+s}{\dot{s}}\dot{u} - u$. Alſo $\dot{s} : \dot{v} :: b+s : \frac{b+s}{\dot{s}}\dot{v} = HC$ or *BL*, therefore *AB* or $x = b + v - \frac{b+s}{\dot{s}}\dot{v}$ where $\dot{s} = \sqrt{\dot{v}^2 + \dot{u}^2}$. If the Evolution begins in *V*, $b = 0$. *Now if the Value of $\dot{v}$ or $\dot{u}$ be got from the Equation of the Curve, and ſubſtituted in theſe Equations, you'll have the Nature of the Curve AM, without any fluctional Quantities. Then* *expunging v, u when it can be done, you'll have the* FIG. *Equation between x and y.* But as s is always found therein, 'tis a Sign that the Nature of the Curve *AM* depends on the Rectification of the Curve *VE*; whoſe Value, when it can be had, may be ſubſtituted in its Room.

EXAMPLE I.

Let AM be a Parabola, $ax = yy$, and, putting $\dot y = 1$, 64.
we ſhall find (by Prob. V.) $CH = \frac{1 + \dot x^2}{\ddot x} = \frac{1}{2}a + 2x$, and $MH = \dot x \times \frac{1 + \dot x^2}{\ddot x} = CH \times \dot x = y + \frac{4xy}{a} = y + \frac{4y^3}{aa}$, and when x and y is 0, $AV = \frac{1}{2}a$. Put $v = VL$, $u = LC$, and we have $v = (AB - AV + BL = x - \frac{1}{2}a + \frac{1}{2}a + 2x =) 3x$, and $u = (MH - MB =) \frac{4y^3}{aa} = \frac{4x\sqrt{ax}}{a}$: Whence $x^3 = \frac{v^3}{27} = \frac{auu}{16}$; or $\frac{16v^3}{27a} = uu$, for the Property of the Curve *VC*, which therefore is a ſemicubical Parabola whoſe Latus rectum is $\frac{27}{16}a$.

Ex. 2.

Let GM be the Logarithmic Curve, $AG = 1$, $AB = x$, 69.
$BM = y$, Subtangent $RB = a$, then $a\dot y = y\dot x$, and (by Prob. V.) $MH = \frac{-aa - yy}{y}$, and $HC = \frac{-aa - yy}{a}$. Let $v = AD = x - \frac{aa + yy}{a}$, $u = DC = 2y + \frac{aa}{y}$, whence $\dot v = \dot x \frac{-2y\dot y}{a} = \frac{aa - 2yy}{ay}\dot y$; and $\dot u = 2\dot y - \frac{aa\dot y}{yy} = \frac{2yy - aa}{yy}\dot y$; therefore $\dot y = \frac{ay\dot v}{aa - 2yy} = \frac{-yy\dot u}{aa - 2yy}$, or $a\dot v$

FIG. $= -y\dot{u}$, the Equation of the Curve wherein the Center C is always found.

Now ſince $\dot{v} = \frac{aa-2yy}{ay}\dot{y}$, therefore v will increaſe when aa is greater then $2yy$, and decreaſe when $2yy$ exceeds aa. Likewiſe u will increaſe when $2yy$ exceeds aa, that is when v decreaſes; and it will decreaſe when v increaſes.

Ex. 3.

65. *Let* AM *be the Cycloid* $AB = x$, $BM = y$, Arch $AD = s$, Axis $TR = c$, $AR = a$, then $y = s + \sqrt{ax-xx}$, $\dot{y} = \sqrt{\frac{a-x}{x}}$, $\ddot{y} = \frac{-a}{2x\sqrt{ax-xx}}$; whence $MH =$ $\frac{1+\dot{y}^2}{-\ddot{y}} = 2\sqrt{ax-xx}$, $CH = \dot{y} \times MH = 2a - 2x$. Let $RN = v$, $NC = u$: then $v = a - x$, and $u =$ $y - 2\sqrt{ax-xx} = s - \sqrt{ax-xx}$, that is (expunging x) $u = s - \sqrt{av-vv}$: And when $x = 0$, $RP = a$. Henc the Curve TCP is alſo a Cycloid equal to the Cycloi AMT. For compleating the Parellelogram $RTQP$ and deſcribing the Semicircle TSQ, $NP = (a - v =$ AB, $ES = (\sqrt{av-vv} =)$ BD, Arch $AD = QS$ and $EC = (c - u = c - s + \sqrt{av-vv} = EN - Q$ $+ ES =)$ $TS + ES$: Therefore TC is the Cycloid.

Ex. 4.

64. *Suppoſe* AM *to be the Catenary*, $AB = x$, $BM =$ $AM = z$, and $z = \sqrt{2ax+xx}$, and $\dot{x} = \frac{z}{a+x}$, $\ddot{x}$ $\frac{aa}{\overline{a+x}^3}$ (putting $\dot{z} = 1$) whence $MH = \frac{\dot{x}\sqrt{1-\dot{x}^2}}{\ddot{x}}$ $\frac{z \times \overline{a+x}}{a}$; and $CH = \frac{1-\dot{x}^2}{\ddot{x}} = a + x$, and when $x =$

HC or *AV* $= a$. Let *VL* $= v$, *LC* $= u$, and we have FIG. 64. $v = 2x$, and $u = \frac{z \times \overline{a+x}}{a} - y = \frac{a+x}{a}\sqrt{2ax+xx} - y$, whence $u = \frac{2a+v}{2a}\sqrt{av+\frac{1}{4}vv} - 2.30258a \times$ Logarithm of $\frac{2a+v+\sqrt{av+\frac{1}{4}vv}}{2a}$, becauſe $y = 2.3025a \times$ into that Log.

Ex. 5.

Suppoſe VM to be a Curve whoſe Tangent MT is = the given Line a, $AB = x$, $BM = y$, $VM = z$, $\dot{y} = 1$, then by the Nature of the Curve $\dot{y} = \frac{-y\dot{x}}{\sqrt{aa-yy}} = \frac{-y\dot{z}}{a}$; hence $\dot{z} = \frac{-a}{y}$, $\ddot{z} = \frac{a}{yy}$: Then $MH = \frac{aa-yy}{-y}$, $HC = -\sqrt{aa-yy}$, the negative Signs ſhow they muſt be taken upward (and to the Right Hand): But in the Vertex *V*, *MH* and *HC* are nothing, therefore the Evolution begins in *V*. 66.

Let *VL* $= v$, *LC* $= u$, then $v = \frac{aa-yy}{y} + y - a = \frac{aa-ay}{y}$, and $y = \frac{aa}{a+v}$: Alſo $u = x + \sqrt{aa-yy}$, and $\dot{u} = \dot{x} - \frac{y\dot{y}}{\sqrt{aa-yy}} = \frac{-\dot{y}}{y}\sqrt{aa-yy} - \frac{y\dot{y}}{\sqrt{aa-yy}} = \frac{-aa\dot{y}}{y\sqrt{aa-yy}} =$ (expunging y and $\dot{y}$) $\frac{a\dot{v}}{\sqrt{2av+vv}}$, an Equation to the Catenary, therefore *VC* is the Catenary Curve.

Ex. 6.

Let AM be the Ciſſoid whoſe Equation is $ax^2 - yx^2 = y^3$, and ſuppoſe $a = 10$. To find the Point *C* of the 70

FIG. 70. Evolute when $y = 2$, $x = 1$. By a former Calculation (Ex. 5. Prob. V.) $\dot{x} = \frac{3xyy + x^3}{2y^3} = \frac{13}{16}$, $\ddot{x} = \frac{6xy + \overline{3xx - 3yy} \times \dot{x}}{2y^3} = \frac{75}{16 \times 16}$; whence (by Prob. V.) $HC = \frac{1 + \dot{x}^2}{\ddot{x}} = 5\frac{2}{3} = BL$, and $MH = \dot{x} \times HC = 4\frac{29}{48}$, and $LC = 2\frac{29}{48}$. And ſo for any other given Point of the Curve.

Ex. 7.

67. *Let BM be the Log. Spiral, TBC* perpendicular to *BM*; and let $c : t : s :: TM : TB : BM$. Then (by Ex. 12. Prob. V.) $MC = \frac{cy}{t}$, and $BC = \frac{sy}{t}$. Therefore ſince the Angle *BCM* is equal *BMT*, the Curve *BC* is alſo a Log. Spiral the ſame with *BM*.

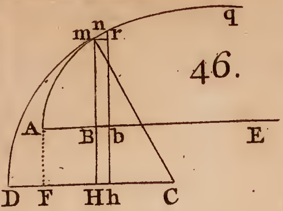
Ex. 8.

68. *Suppoſe BM be an Hyperbolic Spiral, BE* perpendicular to *BM*, *MC* perpendicular to the Tangent in *M*; let Radius $BA = a$, Arch $AO = n$, $BM = y$, then (by Ex. 15. Prob. V.) Radius of Curvature in $M = \frac{EM^3}{yy}$; therefore draw *EG* perpendicular to *EM* and and *GC* perpendicular to *GM*, and the Point *C* is in the Evolute required.

PROB. VIII.

To determine the Length of Curve Lines.

71. Draw the Ordinate *BM* perpendicular to the Axis *AB*, and *nb* parallel and infinitely near to *MB*. And in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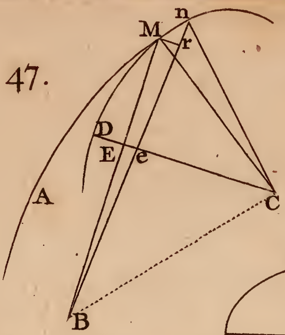
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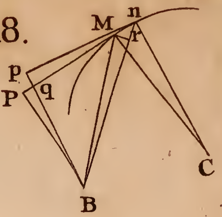
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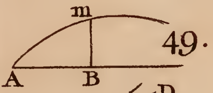
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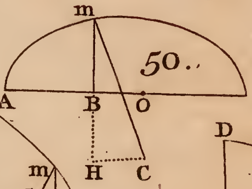
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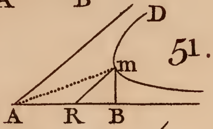
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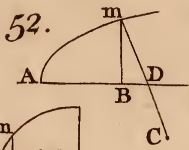
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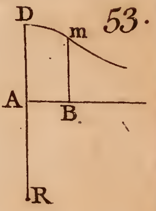
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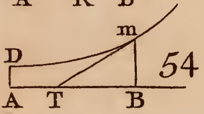
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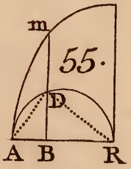
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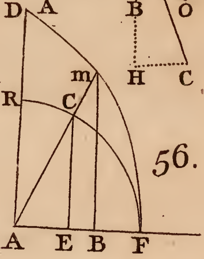
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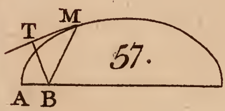
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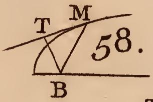
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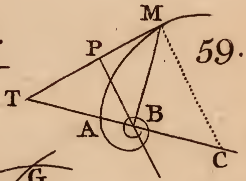
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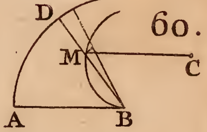
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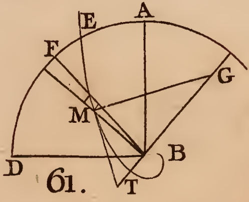
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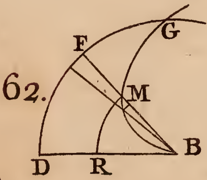
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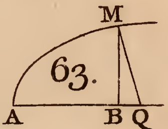
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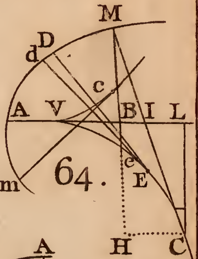
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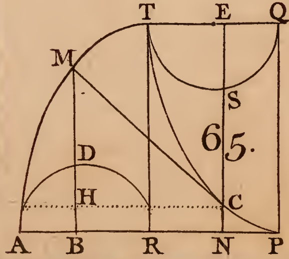
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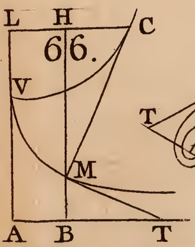
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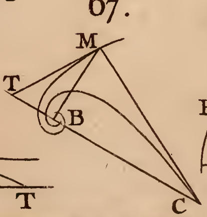
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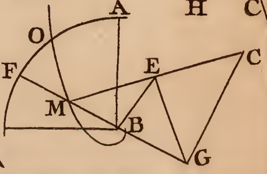
68.

in Spirals draw Bn infinitely near BM; and let Mr be perpendicular to nB or nb. Let the Abſciſſa $AB = x$, Ordinate $BM = y$, Curve $AM = z$, $Mr = \acute{x}$, $rn = \acute{y}$, $Mn = \acute{z}$. In the right angled Triangle Mnr, $\acute{z} = \sqrt{\acute{x}^2 + \acute{y}^2}$, therefore $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$; wherefore FIG. 72.

By the Nature of the Curve exterminate $\dot{x}$ *or* $\dot{y}$ *out of the Equation* $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$; *and the Fluent will give* z *the Length of the Curve.*

EXAMPLE 1.

Let the Nature of the Curve be $\frac{2}{3aa} \times \overline{aa+xx}^{\frac{3}{2}} = y$. 73.

Then its Fluxion is $\dot{y} = \frac{2x\dot{x}}{aa}\sqrt{aa+xx}$: Then $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2} = \dot{x}\sqrt{1 + \frac{4xx}{aa} + \frac{4x^4}{a^4}} = \dot{x} + \frac{2x^2\dot{x}}{aa}$: And (by Form 1ſt.) the Fluent $z = x + \frac{2x^3}{3aa}$, which needs no Correction.

Ex. 2.

In the common Parabola $ax = yy$. Here $\dot{x} = \frac{2y\dot{y}}{a}$, whence $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2} = \frac{\dot{y}}{a}\sqrt{aa+yy}$: And by Form the 9th and 13th, the corrected Fluent $z = \frac{y}{a}\sqrt{\frac{1}{4}aa+yy} + \frac{2.30258a}{4} \times \text{Log.}\ \frac{y + \sqrt{\frac{1}{4}aa+yy}}{\frac{1}{2}a}$. 74.

Ex. 3.

In the ſemicubical Parabola $ax^2 = y^3$, or $x = \frac{y^{\frac{3}{2}}}{a^{\frac{1}{2}}}$, 74.

and

FIG. 74. and $\dot{x} = \frac{3y^{\frac{1}{2}}\dot{y}}{2a^{\frac{1}{2}}}$: And therefore $\dot{z} = \sqrt{\dot{x}^2+\dot{y}^2} =$ $\sqrt{1+\frac{9y}{4a}}$, whoſe Fluent, by Form the 3d, is $z = \frac{8}{2}$ $\times \overline{1+\frac{9y}{4a}}|^{\frac{3}{2}}$. But in the Vertex, y and $z=0$; therefore (Prob. 12. Sect. I.) the Fluent corrected is $z=$ $\frac{8a}{27} \times \overline{1+\frac{9y}{4a}}|^{\frac{3}{2}} - \frac{8a}{27}$.

Ex. 4.

74. *Let $ax^m = y^{m+1}$ be an Equation for various Parabola*

Then $x = \frac{y^{\frac{m+1}{m}}}{a^{\frac{1}{m}}}$, and $\dot{x} = \frac{m+1}{ma^{\frac{1}{m}}} y^{\frac{1}{m}}\dot{y}$, and $\dot{x}^2 =$ $\overline{\frac{m+1}{ma^{\frac{1}{m}}}}|^2 y^{\frac{2}{m}}\dot{y}^2 = by^{\frac{2}{m}}\dot{y}^2$ (putting $b = \overline{\frac{m+1}{ma^{\frac{1}{m}}}}|^2$)

Whence $\dot{z} = \dot{y}\sqrt{1+by^{\frac{2}{m}}} = y^{\frac{1}{m}}\dot{y}\sqrt{b+y^{-\frac{2}{m}}}$

Now the Fluent of $y^{\frac{1}{m}}\dot{y}\sqrt{b+y^{-\frac{2}{m}}}$ will be ha by Form the 15th in finite Terms, when m is an poſitive even Number.

And likewiſe the Fluent of $\dot{y}\sqrt{1+by^{\frac{2}{m}}}$ will b found by Form the 11th, when m is any odd Number; finding firſt the Fluent of $y^{\frac{1}{m}-1}\dot{y}\sqrt{1+by}$ by Form 9th and 13th.

Ex.

Ex. 5.

To find the Length of the Arch of the Circle AM. FIG. 75.
Let Radius $AC=r$, Sine $BM=y$, $AB=x$, by the Nature of the Circle $2rx-xx=yy$, and $\dot{x}=\frac{y\dot{y}}{r-x}=\frac{y\dot{y}}{\sqrt{rr-yy}}$, therefore $\dot{z}=\sqrt{\dot{x}^2+\dot{y}^2}=\frac{r\dot{y}}{\sqrt{rr-yy}}$; and, by Form the 16th, $z=y+\frac{y^2}{2.3.rr}A+\frac{3\cdot 3yy}{4\cdot 5rr}B+\frac{5\cdot 5\cdot yy}{6.7.rr}C$, &c. where A, B, C, &c. are the whole foregoing Terms.

Otherwise thus,

Let $AC=r$, Tangent $AT=t$, $Tt=\acute{t}$, $AM=z$, 76.
$Mn=\acute{z}$, then $CT=\sqrt{rr+tt}$. By the similar Triangles CAT and Ttr, $Tr=\frac{r\acute{t}}{\sqrt{rr+tt}}$; and by the similar Triangles, CMn, CTr, $\acute{z}=\frac{rr\acute{t}}{rr+tt}$, or

$$\dot{z}=\frac{rr\dot{t}}{rr+tt}=\dot{t}-\frac{t^2\dot{t}}{rr}+\frac{t^4\dot{t}}{r^4}-\frac{t^6\dot{t}}{r^6},\quad \text{\&c.}$$

whence (by Form the 16th) the Fluent $z=t-\frac{t^3}{3rr}+\frac{t^5}{5r^4}-\frac{t^7}{7r^6}+\frac{t^9}{9r^8}$, &c.

Now if $r=1$, and Arch $AM=30°$, then $t=\sqrt{\frac{1}{3}}$: And then $z=\sqrt{\frac{1}{3}}\times:1-\frac{1}{3\cdot 3}+\frac{1}{5\cdot 3^2}-\frac{1}{7\cdot 3^3}+\frac{1}{9\cdot 3^4}-$ &c. $=\frac{1}{12}$ the Circumference: And half the Circumference $=\frac{\sqrt{12}}{1}-\frac{\frac{1}{3}A}{3}-\frac{\frac{1}{3}B}{5}-\frac{\frac{1}{3}C}{7}$

FIG. $-\frac{\frac{1}{3}D}{9}$ &c. where A, B, C, &c. are the preceding Numerators with their Signs.

Or thus, by collecting every two Terms into one, we ſhall have half the Circumference $= \frac{16}{3}\sqrt{3} \times$ into $\frac{1}{1.3} + \frac{2}{5.7.9} + \frac{3}{9.11.9^2} + \frac{4}{13.15.9^3}$, &c. that is putting $\frac{16}{3}\sqrt{3} = A$, $\frac{1}{9}A = B$, $\frac{1}{9}B = C$, $\frac{1}{9}C = D$, &c. then half the Circumference $= \frac{1}{1.3}A + \frac{2}{5.7}B + \frac{3}{9.11}C + \frac{4}{13.15}D + \frac{5}{17.19}E + \frac{6}{21.23}F +$ &c. which Series will be as follows.

$$\begin{array}{r}
3{,}079201\ 435678\ 004077\ 38 \\
58651\ 455917\ 676268\ 14 \\
3455\ 893867\ 203147\ 11 \\
259\ 930478\ 900749\ 53 \\
21\ 794996\ 295857\ 48 \\
1\ 943349\ 566200\ 68 \\
180259\ 667580\ 92 \\
17186\ 683703\ 87 \\
1672\ 167602\ 79 \\
165\ 238193\ 86 \\
16\ 530067\ 76 \\
1\ 670177\ 49 \\
170147\ 80 \\
17454\ 05 \\
1801\ 06 \\
186\ 80 \\
19\ 46 \\
2\ 04 \\
22 \\
2 \\
\hline
3{,}141592\ 653589\ 793238\ 46\ \&c.
\end{array}$$

= Half the Circumference.

Ex. 6.

Ex. 6.

Let FMD be the Quadrant of an Ellipſis, AD $=a$, FIG. 77.
AF $=c$, *AB* $=x$, *BM* $=y$, *FM* $=z$. Then by the Nature of the Curve $y=\frac{c}{a}\sqrt{aa-xx}$, and $\dot{y}=\frac{-cx\dot{x}}{a\sqrt{aa-xx}}$; whence $\dot{z}=\sqrt{\dot{x}^2+\dot{y}^2}=\frac{\dot{x}\sqrt{a^4-aaxx+ccxx}}{a\sqrt{aa-xx}}$ $=$(putting $d=\frac{aa-cc}{aa}$) $\frac{\dot{x}\sqrt{aa-dxx}}{\sqrt{aa-xx}}=\frac{\dot{x}\sqrt{\overline{a^2-dxx}\times\overline{aa-xx}}}{aa-xx}$ $=\dot{x}+\frac{1-d}{2aa}x^2\dot{x}+\frac{2-2d-dd}{8a^4}x^4\dot{x}$ &c; whence $z=x+\frac{1-d}{6aa}x^3+\frac{3-2d-dd}{40a^4}x^5+\frac{5-3d-dd-d^3}{112a^6}x^7$ $+$ &c. for the Arch *FM*.

But for the Quadrant *FD*, $\dot{z}=\frac{\dot{x}\sqrt{aa-dxx}}{\sqrt{aa-xx}}=$ $\frac{\dot{x}}{\sqrt{aa-xx}}\times : a-\frac{dxx}{2a}-\frac{ddx^4}{2.4a^3}-\frac{3d^3x^6}{2.4.6a^5}$ &c. but the Fluent of $\frac{\dot{x}}{\sqrt{aa-xx}}=$ Arch of the circumſcribing Quadrant of the Circle divided by $a=\frac{\mathcal{Q}}{a}$; whence the Quadrant *FD* or $z=\mathcal{Q}\times 1-\frac{d}{2.2}+$ $\frac{3d}{4.4}A+\frac{3.5d}{6.6}B+\frac{5.7d}{8.8}C+$ &c, by Form 17th.

Ex. 7.

To find the Length of the Arch of the Hyperbola. 78.
Let $a=$ half the Tranſverſe, $c=AF$ half the conju-

gate,

FIG. 78. gate, $AB = x$, $BM = y$: Then $y = \sqrt{cc + \frac{cc}{aa}xx}$, and $\dot{y} = \frac{cx\dot{x}}{\sqrt{a^4 + aaxx}}$, and $\dot{x}^2 + \dot{y}^2 = \frac{aa + \frac{aa+cc}{aa}xx}{aa + xx}\dot{x}^2$; put $d = \frac{aa+cc}{aa}$, and then $\dot{z} = \dot{x}\sqrt{\frac{aa + dxx}{aa + xx}} = \dot{x}\sqrt{1 - \frac{1-d}{aa}x^2 - \frac{1-d}{a^4}x^4 - \frac{1-d}{a^6}x^6} = \dot{x} \times : 1 - \frac{1-d}{2aa}x^2 + \frac{3-2d-dd}{8a^4}x^4 - \frac{5-3d-dd-d^3}{16a^6}x^6 +$ &c; therefore $z = x - \frac{1-d}{6aa}x^3 + \frac{3-2d-dd}{40a^4}x^5 - \frac{5-3d-dd-d^3}{112a^6}x^7$ &c for the Arch FM.

Ex. 8.

79. *Let AM be the Cycloid*, $AB = x$, $BM = y$, $AR = a$, $BD = v$, Arch $AD = s$, then $y = s + v$ By the Nature of the Circle $vv = ax - xx$, and $\dot{v} = \frac{a\dot{x} - 2x\dot{x}}{2v}$, and $\dot{s} = \frac{a\dot{x}}{2v}$; therefore $\dot{y} = \dot{s} + \dot{v} = \frac{a - x}{v}\dot{x}$, whence $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2} = \dot{x}\sqrt{1 + \frac{\overline{a-x}^2}{vv}} = \frac{\dot{x}}{v}\sqrt{aa - ax} = \frac{\dot{x}\sqrt{aa - ax}}{\sqrt{ax - xx}} = \dot{x}\sqrt{\frac{a}{x}}$. Therefore $z = 2\sqrt{ax} = 2$ Cord AD.

Ex. 9.

69. *Let GM be the Log. Curve*, $AB = x$; $BM = y$, $AG = b$, Subtangent $BR = a$, then $a\dot{y} = y\dot{x}$, therefore $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2} = \frac{\dot{y}}{y}\sqrt{aa + yy} = \frac{y\dot{y}}{\sqrt{aa + yy}} +$

$aay^{-2}\dot{y}$

$\frac{aay^{-2}\dot{y}}{\sqrt{1+aay^{-2}}}$. Whence (by Form the 3d and 9th) FIG. 69.
$z=\sqrt{aa+yy}-2.30258a\times\text{Log.}\,\frac{a+\sqrt{aa+yy}}{y}$;
and corrected (by Prop. XII.) z or $GM=\sqrt{aa+yy}$
$-\sqrt{aa+bb}+2.302585a\times\text{Log.}\,\frac{ay+y\sqrt{aa+bb}}{ab+b\sqrt{aa+yy}}$.

Ex. 10.

In the Ciſſoid DM, let $AD=a$, $AB=x$, $BM=y$, 80.
then $y=\frac{\overline{a-x}^{2}}{\sqrt{ax-xx}}=\frac{\overline{a-x}|^{\frac{3}{2}}}{x^{\frac{1}{2}}}$, and $\dot{y}=$
$\frac{-a\dot{x}-2x\dot{x}}{2xx}\sqrt{ax-xx}$, whence $\dot{z}=\sqrt{\dot{x}^{2}+\dot{y}^{2}}$
$=\frac{a\dot{x}}{2x}\sqrt{\frac{a+3x}{x}}=\frac{ax^{-\frac{3}{2}}\dot{x}}{2}\sqrt{a+3x}$, and
by Form the 9th, 13th and 12th) $z=2.3025a\sqrt{3}$
$\times$ Log. $\sqrt{3x}+\sqrt{a+3x}:-a\sqrt{\frac{a+3x}{x}}$. And
he Fluent corrected, z or $MD=a\sqrt{\frac{a+3x}{x}}-2a$
$+\,2.302585a\sqrt{3}\times\text{Log.}\,\frac{\sqrt{3a}+\sqrt{4a}}{\sqrt{3x}+\sqrt{a+3x}}$.

Ex. 11.

Let AM be the Quadratix, $AB=x$, $BM=y$, Arch 81.
$K=t$, Sine $KG=v$, $AC=a$. Then (by Ex. 5th)
$=v+\frac{v^{3}}{6aa}+\frac{3v^{5}}{40a^{4}}+\frac{5v^{7}}{112a^{6}}+$ &c. $=y$, by
he Nature of the Quadratrix. Then by Reverſion
f Series $v=y-\frac{y^{3}}{6aa}+\frac{y^{5}}{120a^{4}}-\frac{y^{7}}{5040a^{6}}$ &c.

FIG. 81. then $\sqrt{aa-vv}=a-\frac{yy}{2a}+\frac{y^4}{24a^3}-\frac{y^6}{720a^5}$ &c. $=CG$; and by ſimilar Triangles (*KG*) $v : GC ::$ (*MB*) y : (*BC*) $a-x=a-\frac{y^2}{3a}-\frac{y^4}{45a^3}-\frac{2y^6}{945a^5}$ &c. whence $x=\frac{yy}{3a}+\frac{y^4}{45a^3}+\frac{2y^6}{945a^5}$ &c. and $\dot{x}=\frac{2y\dot{y}}{3a}+\frac{4y^3\dot{y}}{45a^3}+\frac{4y^5\dot{y}}{315a^5}$ &c. therefore $\dot{z}=\sqrt{\dot{x}^2+\dot{y}^2}=\dot{y}\times : 1+\frac{2y^2}{9aa}+\frac{14y^4}{405a^4}+\frac{604y^6}{127575a^6}$ &c. and $z=y+\frac{2y^3}{27a^2}+\frac{14y^5}{2025a^4}+\frac{604y^7}{893025a^6}$ &c.

Ex. 12.

82. *Let BM be Archimedes's Spiral, whoſe Equation is* $rv=cy$, putting $v=$ Arch *ED*. Then $r\dot{v}=c\dot{y}$, but (by ſimilar Triangles) $\dot{v}=\frac{r\dot{x}}{y}$, whence $\dot{x}=\frac{cy\dot{y}}{rr}$: Therefore $\dot{z}=\sqrt{\dot{x}^2+\dot{y}^2}=\frac{\dot{y}}{rr}\times\sqrt{r^4+ccyy}$; and (by Form 9th and 13th) the Fluent (corrected) is $z=\frac{y}{2rr}\sqrt{r^4+ccyy}+\frac{2.302585rr}{2c}\times\text{Log.}\frac{cy+\sqrt{r^4+ccyy}}{rr}$.

Ex. 13.

82. *Let BM be the Log. Spiral,* and let c, t, s expreſs the Ratio of the Tangent *TM*, Subtangent *TB*, and Ordinate *BM*. Then $\dot{z}=\frac{c\dot{y}}{s}$ and the Fluent $z=\frac{cy}{s}=TM$. Therefore the Length of the Spiral *MAB* making an infinite Number of Revolutions is equal to the Tangent *MT*.

Ex. 14.

Ex. 14.

FIG. 83.

Let BM be the hyperbolic Spiral, $AB=a$, Arch $AO=b$, Arch $AD=v$, $MB=y$, then $ab=vy$, and thence $v\dot{y}+y\dot{v}=0$, and $y\dot{v}=-v\dot{y}=\frac{-ab\dot{y}}{y}$; and by ſimilar Triangles $-y\dot{v}=a\dot{x}$, and thence $\dot{x}=\frac{b\dot{y}}{y}$; therefore $\dot{z}=\sqrt{\dot{x}^2+\dot{y}^2}=\frac{\dot{y}}{y}\sqrt{bb+yy}=\frac{bby^{-2}\dot{y}}{\sqrt{1+bby^{-2}}}+\frac{y\dot{y}}{\sqrt{bb+yy}}$. And the Fluent (by Form the 3d and 9th) is $z=\sqrt{bb+yy}-b\times 2.302585\ \text{Log.}\ \frac{b+\sqrt{bb+yy}}{y}$: Whence taking any Ordinate $BQ=d$, the Length of the Arch $QM=\sqrt{bb+dd}-\sqrt{bb+yy}-2.30258b\times\text{Log.}\ \frac{by+y\sqrt{bb+dd}}{bd+d\sqrt{bb+yy}}$.

PROB. IX.

To transform Spirals into geometrical Curves, or geometrical Curves into Spirals of equal Length.

84.

Let ADd be a geometrical Curve related to the Axis AB; let the perpendicular Ordinates BD, bd, be infinitely near together; and let the Points B, b be ſuppoſed to approach one another, and to coincide in B, whilſt the Particle of the Curve Dd remains the ſame; then the Parallelogram $BDdb$ will be changed into the Triangle BDd. In like Manner let all

FIG. 84. all the Points in *AB* be ſuppoſed to be contracted into one Point *B*, and then the Figure *ADd* will be transform'd into the Spiral *QDd*, whoſe Center is *B*, in which the correſponding Ordinates are equal, and the Triangles *Ddr* in both Figures ſimilar and equal; and the Area of the Figure *ABD* twice the Area of the Spiral *BQD*.

85. Again let *ADd* be a Spiral, *ABD* its Complement; *BD*, *bd*, two Arches of Circles infinitely near each other, whoſe Center is *A*; *AH* a geometrical Curve (to the Abſciſſa $AB = AD$) equal in Length to the Spiral: Let *BH*, *bh* be drawn infinitely near each other, and *HK* parallel to *AB*. Then the Triangles *DdC* and *HbK* are ſimilar and equal, therefore $dC = Kh$. Now let *AB* or $AD = y$, $BH = u$, Arch $BD = v$, and by ſimilar Triangles $y : v :: Ab : bC :: Bb : bC - v$ $= \frac{v}{y} \times Bb$. Therefore $bC + CD - v$ or $bd - v = Kb$ $+ \frac{v}{y} \times Bb$: Therefore (becauſe the Moments are as the Fluxions) $\dot{v} = \dot{u} + \frac{v\dot{y}}{y}$.

84. 1. *Therefore in Curves referr'd to an Axis, let* $AB = x$, $BD = y$, *Radius BR or* $BT = b$, *Arch* $RT = v$, $Dr = \dot{x}$, $rd = \dot{y}$, $TS = \dot{v}$. *In the Curve AD, expunge x* and $\dot{x}$ *out of the Equation of the Curve, by Help of the Equation* $b\dot{x} = y\dot{v}$; *and the Fluent will give the Nature of the Spiral. Likewiſe in the Spiral QD, expunge all Quantities except x and y and their Fluxions out of the Equation* $b\dot{x} = y\dot{v}$, *by Help of the Equation of the Curve; and the Fluent will ſhow the Nature of the geometrical Curve AD.*

85. 2. *In the Spiral AD, where there is given the Relation of AD (y) to the Arch BD (v): By help of the Equation of the Spiral, exterminate v and* $\dot{v}$ *out of the Equation*

Equation $\dot v = \dot u + \frac{v\dot y}{y}$, and the Fluent gives the Na-ure of the Curve AH. Or by the Equation of the Curve AH, expunge u and $\dot u$ out of the Equation $\dot v = \dot u + \frac{v\dot y}{y}$, and the Fluent gives the Nature of the Spiral. FIG. 85.

EXAMPLE 1.

Let ax=yy denote a Parabola ADd, then we have $\dot x = \frac{2y\dot y}{a} = \frac{y\dot v}{b}$, and $2b\dot y = a\dot v$; whence the Fluent $2by = av$; therefore the Spiral *QD* is that of *Archimedes.* 84.

Ex. 2.

Let AD be a Circle, $2ax - xx = yy$, Arch $AD = z$, therefore $\dot x = \frac{y\dot y}{a-x} = \frac{y\dot y}{\sqrt{aa-yy}} = \frac{y\dot v}{b}$, therefore $\dot v = \frac{b\dot y}{\sqrt{aa-yy}}$, let $b = a$, then $\dot v = \frac{a\dot y}{\sqrt{aa-yy}} = \dot z$ herefore $v = z$, that is $RT = AD$, and $BD = BD = TF$; draw *GD*, then ſince $BG = BT$, and $BD = TF$, nd $\angle DBG = \angle FTB$, therefore $\angle GDB = \angle BFT =$ right Angle, and the Curve *BDG* is a Semicircle. 86.

Ex. 3.

Let aa=xy be the Equation of an equilateral Hyper-ola between the Aſſymptotes, then $x\dot y + y\dot x = 0$, and $\dot x = \frac{-x\dot y}{y} = -\frac{aa}{yy}\dot y = \frac{y\dot v}{b}$. Therefore $-\frac{baa\dot y}{y^3} = \dot v$, and thence $v = \frac{baa}{2yy}$, or putting $b = 2a$, $^3 = vyy$, for the Nature of the Spiral *QDd*. 87.

Ex. 4.

Ex. 4.

FIG. 88. *Let ED be the Logarithmic Curve*, whoſe Equation is $y\dot{x} = a\dot{y}$, then $\dot{x} = \frac{a\dot{y}}{y} = \frac{y\dot{v}}{b}$, and $\frac{ab\dot{y}}{yy} = \dot{v}$, therefore $v = -\frac{ab}{y}$: Therefore any Portion of the Arch *RT* or v is $= ab \times \frac{DB - EA}{DB \times EA}$; *BD*, *EA* being the correſponding Ordinates.

Ex. 5.

Let $ay = bx$ *denote a plain Triangle*, then $a\dot{y} = b\dot{x} = y\dot{v}$, therefore $\frac{a\dot{y}}{y} = \dot{v}$, conſequently $v = a \times \text{Log. } y$, an Equation to the Log. Spiral.

Ex. 6.

84. *Let QD be the Spiral of Archimeds*, whoſe Equation is $dy = av$; then $\frac{d\dot{y}}{a} = \dot{v} = \frac{b\dot{x}}{y}$, or $dy\dot{y} = ab\dot{x}$, and the Fluent $\frac{dyy}{z} = abx$, or $yy = \frac{2ab}{d}x$, an Equation to a Parabola.

85. Or thus: *Let* $AB = y$, $BD = v$, then $av = yy$, whence $\frac{2y\dot{y}}{a} = \dot{v} = \dot{u} + \frac{v\dot{y}}{y} = \dot{u} + \frac{y\dot{y}}{a}$, and $\dot{u} = \frac{y\dot{y}}{a}$, therefore $2au = yy$, an Equation to the common Parabola, as before.

Ex. 7.

84. *Let QD be the Log. Spiral*, whoſe Equation is $a\dot{y} = b\dot{x}$, therefore $ay = bx$ a plain Triangle.

Ex. 8.

Ex. 8.

Let QD be the Hyperbolic Spiral, then $ab = vy$, and $v\dot{y} + y\dot{v} = 0$, or $\dot{v} = \frac{-v\dot{y}}{y} = \frac{-ab\dot{y}}{yy} = \frac{b\dot{x}}{y}$, or $\frac{-a\dot{y}}{y} = \dot{x}$, whence the Fluent $\frac{-x}{a} = \text{Log. } y$, an Equation to the Log. Curve. FIG. 88.

Ex. 9.

Suppoſe $y^3 = av^2$ *to denote the Spiral ADd,* then $\dot{v} = \frac{3y^{\frac{1}{2}}\dot{y}}{2\sqrt{a}} = \dot{u} + \frac{v\dot{y}}{y} = \dot{u} + \frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{a}}$; therefore $\dot{u} = \frac{y^{\frac{1}{2}}\dot{y}}{2\sqrt{a}}$, whence $u = \frac{y^{\frac{3}{2}}}{3\sqrt{a}}$, or $9au^2 = y^3$, a ſemicubical Parabola. 85.

Ex. 10.

If $v = y\sqrt{\frac{a+y}{c}}$ *be the Equation of the Spiral;* then $\dot{v} = \frac{2a\dot{y} + 3y\dot{y}}{2\sqrt{ca+cy}} = \dot{u} + \frac{v\dot{y}}{y} = \dot{u} + \dot{y}\sqrt{\frac{a+y}{c}} = \dot{u} + \frac{a\dot{y} + y\dot{y}}{\sqrt{ca+cy}}$; whence $\dot{u} = \frac{y\dot{y}}{2\sqrt{ca+cy}}$, therefore $u = \frac{y-2a}{3c}\sqrt{ca+cy}$. 85.

Ex. 11.

Let BDdE be a Semicircle, $BD = y$, $BE = a$, by ſimilar Triangles $Dr = \frac{BD \times rd}{ED}$, or $\dot{x} = \frac{y\dot{y}}{\sqrt{aa-yy}}$, whence $x = -\sqrt{aa-yy}$; when corrected $x = a - \sqrt{aa-yy} = BE - ED$, therefore *ADL* is the Quadrant of a Circle, whoſe Radius is *a*. 89.

Ex. 12.

FIG. 90. *Let QDd be a Parabola, B* the Focus, $a =$ Latus Rectûm, $BD = y$, DP a Tangent: By the Nature of the Figure the Perpendicular $BP = \frac{1}{2}\sqrt{ay}$, and by ſimilar Triangles $Dr = \frac{BP \times dr}{PD}$ or $\dot{x} = \frac{a\dot{y}}{\sqrt{4ay - aa}}$; therefore $x = \sqrt{ay - \frac{1}{4}aa} =$ Ordinate DI. Hence the Curve will be the ſame Parabola referr'd to the Axis AB; making $AQ = QB$, and AB perpendicular to AQ.

PROB. X.

To find the Areas of Curves.

91. In any Curve AD related to the Abſciſſa AB, produce the Ordinate DB till BE be equal to a given Line, and compleating the Parallelogram $ABEC$: If the Arears $ACEB$ and ADB be conceived to be generated by the right Lines BE, BD, moving along the Abſciſſa AB; then the Fluxions of the Areas $ACEB$ and ABD will be as the deſcribing Lines BE, BD drawn into their Velocities of moving, that i into the Fluxions of the Abſciſſas: Now ſince $BE \times AB =$ Area $ACEB$, and therefore the Fluxion of the Area $ACEB = BE \times$ Fluxion of AB; conſequently the Fluxion of the Area $ABD = DB \times$ Fluxion o AB.

92. In like Manner in Curves related to a fixt Point B, the fluxionary Triangle BDM is the Moment of the Area, and this is $= BM \times \frac{DK}{2}$, or = the Perpen-

dicula

dicular *BT* (on the Tangent at *M*) × by $\frac{1}{2}DM$; but *DR* and *DM* are as their generating Fluxions. Hence

1. *In Curves related to an Axis AB, let* $AB = x$, $BD = y$, *Area* $ABD = z$, *then* $\dot{z} = y\dot{x}$; *therefore by the Equation of the Curve, expunge one of the Quantities y or* $\dot{x}$ *out of the Equation* $\dot{z} = y\dot{x}$, *and finding the Fluent it gives the Value of z the Area. Sometimes it will be neceſſary to find the Area of the Complement ABD, but the Rule will ſtill be the ſame if you make AB the Abſciſſa.* 91.

Note, If the Ordinates are not at right Angles to the Abſciſſa, the Area before found muſt be diminiſhed in the Ratio of Radius to the Sine of the true Angle.

2. *In Curves related to a fixt Point B, let* $BD = y$, *Perpendicular BT (on the Tangent)* $= p$, *Curve* $AD = v$, $Dr = \dot{x}$. *Area* $BAD = z$: *Then by the Nature of the Curve expunge y or* $\dot{x}$ *out of the Equation* $\dot{z} = \frac{y\dot{x}}{2}$; *or expunge p or* $\dot{v}$ *out of the Equation* $\dot{z} = \frac{p\dot{v}}{2}$, *and the Fluent will give z the Area.* 92.

And if the fluxionary Triangle BDM can be computed any other Way, and the heterogeneous Quantities expung'd by the Equation of the Curve; the Fluent will give the Area as before.

EXAMPLE I.

To find the Area of a Triangle. Let the Baſe $CD = b$, Perpendicular $AQ = p$, $AB = x$, dD (parallel to CD) $= y$: By ſimilar Triangles $y = \frac{bx}{p}$, therefore $\dot{z} = y\dot{x} = \frac{bx\dot{x}}{p}$, therefore the Fluent $z = \frac{bxx}{2p}$, $= \frac{xy}{2}$, and when $x = p$, and $y = b$, the Area $z =$ 93.

FIG. $\frac{bp}{2}$. Or ſince $\dot{x} = \frac{p\dot{y}}{b}$, therefore $\dot{z} = y\dot{x} = \frac{py\dot{y}}{b}$, and $z = \frac{pyy}{2b} = \frac{xy}{2}$, as before.

Otherwiſe thus:

94. Let the Perpendicular $BC = p$, $BD = y$, $de = \acute{y}$; and by ſimilar Triangles $Dd = \frac{y\acute{y}}{\sqrt{yy - pp}}$, therefore $\acute{z} = \frac{py\acute{y}}{2\sqrt{yy - pp}} =$ Area BDd, and $\dot{z} = \frac{py\dot{y}}{2\sqrt{yy - pp}}$, and thence $z = \frac{1}{2}p\sqrt{yy - pp}$. Now in the Point C, y is a Minimum $= p$. Therefore by the Schol. Prop. XII. firſt the Part ABC muſt be found, which is $\frac{p}{2}\sqrt{AB^2 - pp}$, and then the Part BCD, which will be $\frac{p}{2} \times \sqrt{BD^2 - pp}$. And the whole $ABD = \frac{p}{2}\sqrt{AB^2 - pp} + \frac{p}{2}\sqrt{BD^2 - pp}$, or $\frac{1}{2}AD \times CB$.

Ex. 2.

91. *Let* ABD *be any Parabola, where* $x = y^m$; then $\dot{x} = my^{m-1}\dot{y}$, therefore $\dot{z} = y\dot{x} = my^m\dot{y}$; and $z = \frac{m}{m+1}y^{m+1} = \frac{m}{m+1}yx$, and if $m = 2$, $z = \frac{2}{3}xy$.

Or thus: Let $Ab = x$, $bD = y$, $x = y^{\frac{1}{m}}$, then $\dot{x} = \frac{1}{m}y^{\frac{1}{m}-1}\dot{y}$, therefore $\dot{z} = y\dot{x} = \frac{1}{m}y^{\frac{1}{m}}\dot{y}$, and thence

$z =$

$z = \frac{1}{m+1} y^{\frac{1}{m}-1} = \frac{1}{m+1} xy$: And if $m=2$, $z=$ FIG.
$\frac{1}{3}xy$, the Area *AbD*.

Ex. 3.

Let FD be an Hyperbola, $CA=a$, $AF=b$, $AB=x$, 95.
$BD=\frac{ab}{a+x}=y$, Then $\dot{z}=y\dot{x}=\frac{ab\dot{x}}{a+x}=b\dot{x}-\frac{bx\dot{x}}{a}+\frac{bx^2\dot{x}}{aa}-\frac{bx^3\dot{x}}{a^3}$ &c. and the Area *ABDF* or $z=bx-\frac{bx^2}{2a}+\frac{bx^3}{3aa}-\frac{bx^4}{4a^3}+\frac{bx^5}{5a^4}$, &c.
Or (by Form 4th) the Fluent (corrected) is $z= 2.302585ab \times \text{Log.} \frac{a+x}{a}$.

Ex. 4.

Let FD be any kind of Hyperbola, $CB=x$, $BD=y$, 95.
and ſuppoſe $y=\frac{1}{x^n}$, then $\dot{z}=y\dot{x}=x^{-n}\dot{x}$; and the Area $z=\frac{1}{1-n}x^{1-n}=\frac{1}{1-n}xy$, for the Area *CBD*.
If the Angle *ABD* is oblique, that Area muſt be diminiſh'd in the Ratio of Radius to the Sine of *ABD*.

If $n=1$ the Space will be infinite, if n be greater than 1 you get the Space *BDE*.

Ex. 5.

Let AD be a Circle, $AC=a$, $AB=x$, $BD=y=$ 96.
$\sqrt{ax-xx}$, then $\dot{z}=y\dot{x}=\dot{x}\sqrt{ax-xx}$, whoſe Fluent (by Form 16th) is $z=\sqrt{ax}\times : \frac{2}{3}x-\frac{x^2}{5a}-\frac{x^3}{4.7a^2}-\frac{1.3x^4}{4.6.9a^3}-\frac{3.5.x^5}{4.6.8.11a^4}$ &c. $=\frac{2x\sqrt{ax}}{3}-\frac{\frac{x}{2a}A}{7}$

+

FIG. 96. $+\frac{\frac{x}{4a}B}{7}+\frac{\frac{3x}{6a}C}{9}+\frac{\frac{5x}{8a}D}{11}$ &c. where A, B, C, &c. are the preceding Numerators. Or (by Form 10, 11, 13) $z=-\frac{aa}{8}\varphi+\frac{2x-a}{4}\sqrt{ax-xx}$ (putting φ $=,01745\times$ twice the Degrees in the Arch whoſe Sine is $\sqrt{\frac{x}{a}}$, and Radius 1.)

If AD be 60°, then $x=\frac{1}{4}a$, and Area $ABD=\frac{1}{12}aa$ $-\frac{3}{2.4.5}A+\frac{1.5}{4.4.7}B+\frac{3.7}{6.4.9}C+\frac{5.9}{8.4.11}D$, to which adding the Triangle $DBE=\frac{aa}{32}\sqrt{3}$, you have $\frac{1}{6}$ the Area of the Circle. Here A, B, C, &c. are the preceding Terms.

If $y=\sqrt{ax+xx}$ be an Equation to a right-angled Hyperbola, we ſhall in like Manner find $z=\sqrt{ax}$ $\times:\frac{2}{3}x+\frac{x^2}{5a}-\frac{x^3}{4.7aa}+\frac{3\cdot x^4}{4.6.9a^3}-\frac{3.5x^5}{4.6.8.11a^4}$ + &c. or $z=\frac{a+2x}{4}\sqrt{ax+xx}-\frac{aa}{8}\varphi$, where $\varphi=2.30258\,\mathrm{Log.}\,\frac{a+2x+2\sqrt{ax+xx}}{a}$.

Likewiſe the Area of an Ellipſis is found after the ſame Manner as the Circle. Or if you put c for the Conjugate, tis no more than multiplying the former Area of the Circle by $\frac{c}{a}$. And thus $\frac{c}{a}$ multiply'd into the former Area of the Hyperbola, gives the Area of the Hyperbola whoſe Conjugate is c.

Ex. 6.

97. *In the Ellipſis FDE*, to find the Area adjoining to the Center A. Let $AE=a$, $AF=c$, $AB=x$, $BD=y$, then $y=\frac{c}{a}\sqrt{aa-xx}$, by the Nature of the Figure;

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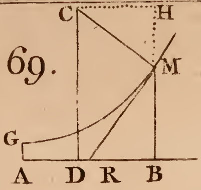

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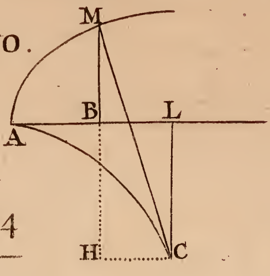

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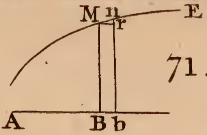

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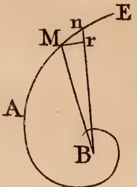

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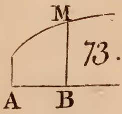

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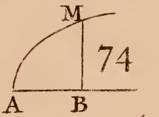

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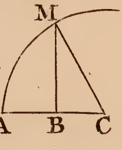

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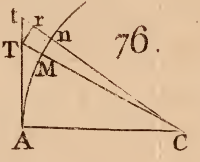

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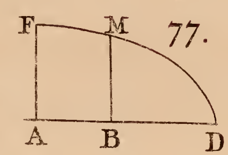

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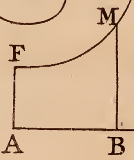

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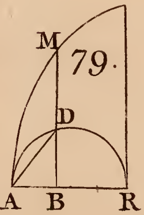

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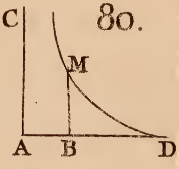

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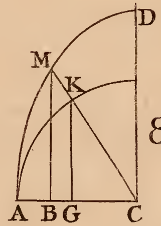

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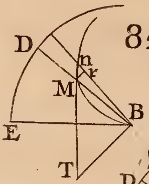

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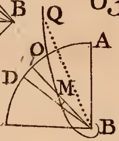

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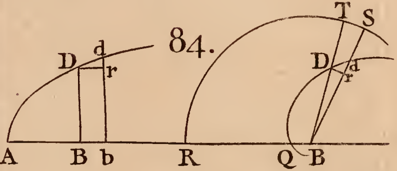

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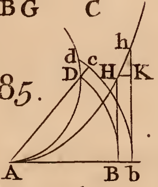

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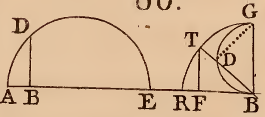

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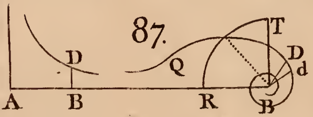

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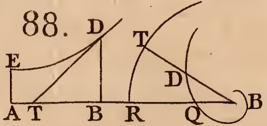

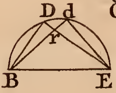

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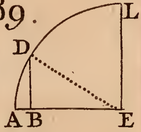

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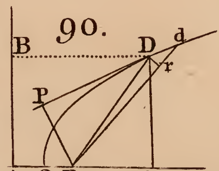

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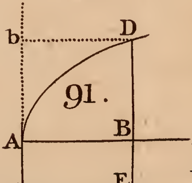

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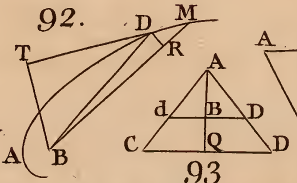

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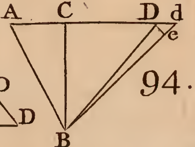

gure; then $\dot{z} = y\dot{x} = \frac{c\dot{x}}{a}\sqrt{aa-xx}$, whence (by Form 16) FIG. 97.

$$z = \frac{cx}{1} - \frac{\frac{xx}{2aa}A}{3} + \frac{\frac{xx}{4aa}B}{5} + \frac{\frac{3xx}{6aa}C}{7} + \frac{\frac{5xx}{8aa}D}{9}$$

&c. for the Area *ABDF*. Or $z = \frac{ca\varphi}{2} + \frac{x}{2a}\sqrt{aa-xx}$, putting $\varphi = ,017453 \times$ Degrees in the Arch whoſe Sine is $\frac{x}{a}$ and Radius 1.

After the ſame Manner in the Hyperbola, when 98.

$y = \frac{c}{a}\sqrt{aa+xx}$, we find $z = \frac{cx}{1} + \frac{\frac{xx}{2aa}A}{3} - \frac{\frac{xx}{4aa}B}{5} - \frac{\frac{3xx}{6aa}C}{7} - \frac{\frac{5xx}{8aa}D}{9}$ &c. Or $z = \frac{ca\varphi}{2} + \frac{cx}{2a}\sqrt{aa+xx}$, putting $\varphi = 2.30258$ Log. $\frac{x+\sqrt{aa+xx}}{a}$.

Ex. 7.

To find the Area ADB of the Ellipſis ADE, generated by the Line BD revolving round the Focus B. 99.

Let $2a$ = Tranſverſe *AE*, $2c$ = Conjugate, *BD* $= y$, draw the Tangent *DT* and *BT* perpendicular to it; then by the conic Sections $BT = \frac{cy}{\sqrt{2ay-yy}}$, whence $DT = \frac{y\sqrt{2ay-yy-cc}}{\sqrt{2ay-yy}}$. And by ſimilar Triangles $DT : TB :: MR : RD$, that is $\sqrt{2ay-yy-cc} : c :: \dot{y} : \dot{x} = \frac{c\dot{y}}{\sqrt{2ay-yy-cc}}$;

whence

FIG. 99. whence $\dot{z} = \frac{y\dot{x}}{2} = \frac{cy\dot{y}}{2\sqrt{-cc+2ay-yy}}$: Pu $v = a - y$, $p = aa - cc$; and (by Form 27th) $\dot{z} =$ $\frac{cv\dot{v}}{2\sqrt{p-vv}} - \frac{ca\dot{v}}{2\sqrt{p-vv}}$; whence (by Forms 3d and 10th) $z = \frac{-c}{2}\sqrt{p-vv} - \frac{ca}{2} \times ,01745$ Degrees in the Arch whoſe Sine is $\sqrt{\frac{vv}{p}}$, but whe $z = 0$, $y = a - \sqrt{p}$, therefore correcting the Fluent, and putting $\varphi =$ Number of Degrees in the Arch whoſe Co-ſine is $\frac{a-y}{\sqrt{aa-cc}}$, and Radius 1, then we have the Area *BAD* or $z = \frac{,0174533ca\varphi}{2} - \frac{c}{2}$ $\sqrt{-cc+2ay-yy}$. And the Area of the Semi-ellipſis $AME = ,01745ca \times 90 = \frac{1}{2}ca \times 3.141592$.

Ex. 8.

100. *To find the parabolic Area BAD, generated by th Line BD revolving round the Focus B.*

Let $a =$ Latus Rectum, $BD = y$; by the Nature of the Parabola the Perpendicular $BT = \frac{1}{2}\sqrt{ay} = p$, and by ſimilar Triangles $DT : DB :: MR : MD$, or $\sqrt{yy-\frac{1}{4}ay} : y :: \dot{y} : \dot{v} = \frac{y\dot{y}}{\sqrt{yy-\frac{1}{4}ay}}$; whence $\dot{z}$ $= \frac{p\dot{v}}{2} = \frac{\sqrt{a}}{4} \times \frac{y\dot{y}}{\sqrt{y-\frac{1}{4}a}}$; and (by Form 3d and 11th) the Fluent $z = \frac{a+2y}{12}\sqrt{ay-\frac{1}{4}aa}$.

Ex. 9.

101. *Let AD be an Hyperbola, B* the Center, $BA = a$, Semi-conjugate $= b$, $BC = x$, Tangent $AT = t$, Or-

dinate $CD=y$; by ſimilar Triangles $ay=tx$, and by the Nature of the Figure $bbxx-bbaa=aayy=ttxx$, and $xx=\frac{bbaa}{bb-tt}$. The fluxionary Triangle BTt is to the fluxionary Triangle BDd, as BT^2 to BD^2 or aa to xx, that is $\frac{a\dot{t}}{2}:\dot{z}::aa:\frac{bbaa}{bb-tt}::bb-tt:bb$; whence $\dot{z}=\frac{\frac{1}{2}bba\dot{t}}{bb-tt}$, and the Fluent $z=\frac{at}{2}+\frac{at^3}{6bb}+\frac{at^5}{10b^4}+\frac{at^7}{14b^6}$ &c. Or (by Form 6th) the Fluent $z=\frac{1}{4}ab\times 2.30258\ \mathrm{Log.}\frac{b+t}{b-t}$ for the hyperbolic Sector BAD.

FIG. 101.

Or thus: Since $t=(\frac{ay}{x}=\frac{ay}{\sqrt{aa+\frac{aa}{bb}yy}}=\frac{by}{\sqrt{bb+yy}}$, therefore $\dot{t}=\frac{b^3\dot{y}}{\overline{bb+yy}^{\frac{3}{2}}}$, therefore $\dot{z}=(\frac{\frac{1}{2}abb\dot{t}}{bb-tt}=)\frac{ab\dot{y}}{2\sqrt{bb+yy}}$; and the Fluent $z=\sqrt{bb+yy}\times:\frac{ay}{2b}-\frac{2yy}{3bb}A-\frac{4yy}{5bb}B-\frac{6yy}{7bb}C$ &c. by Form the 15th. Or (by Form the 9th) $z=\frac{1}{2}ab\times 2.30258\times\mathrm{Log.}\frac{y+\sqrt{bb+yy}}{b}=$ Sector BAD.

Ex. 10.

Let RD be the Logarithmic Curve, $AB=x$, $BD=y$, $AR=b$, Subtangent $=a$, by ſimilar Triangles $a\dot{y}=\dot{x}=\dot{z}$, whence $z=ay$, and (corrected) Area $ABDR=ay-ab$. 102.

Ex. 11.

FIG. 103. *Let AD be the Ciſſoid of Diocles,* $AE = a$, $AB = x$, $BD = y$, by the Nature of the Curve $y = \frac{x^{\frac{3}{2}}}{\sqrt{a-x}}$; therefore $\dot{z} = y\dot{x} = x^{\frac{3}{2}}\dot{x} \times \overline{a-x}|^{-\frac{1}{2}}$, and the Fluent (by Form the 10th and 11th) is $z = 3$ Sectors $\overline{CAM - \frac{3}{4}a - \frac{1}{2}x}\sqrt{ax-xx}$, that is $z = 3AMB - 2x\sqrt{ax-xx}$.

Ex. 12.

104. *Let ED be the Concoid of Nichomedes,* $CA = b$, $AE = a$, $AB = x$, $BD = y$; the Equation of the Curve is $\overline{b+y}\sqrt{aa-yy} = xy$, in Fluxions $\dot{y}\sqrt{aa-yy} - \overline{b+y} \times y\dot{y} \times \overline{aa-yy}|^{-\frac{1}{2}} = x\dot{y} + y\dot{x}$, therefore $\dot{z} = y\dot{x} = \dot{y}\sqrt{aa-yy} - y\dot{y} \times \frac{b+y}{\sqrt{aa-yy}}$ $(-x\dot{y} =) - \frac{b+y}{y} \times \dot{y}\sqrt{aa-yy} = \frac{-y^2\dot{y}}{\sqrt{aa-yy}} - \frac{baa\dot{y}}{y\sqrt{aa-yy}} = \frac{-y^2\dot{y}}{\sqrt{aa-yy}} - \frac{ba^2y^{-2}\dot{y}}{\sqrt{-1+a^2y^{-2}}}$; whoſe corrected Fluent (by Form 10 and 11, and Form 9th) is $z = \frac{1}{2}y\sqrt{aa-yy} + 2.30258ab \times \text{Log.}\ \frac{a+\sqrt{aa-yy}}{y}$ $+ ,017453$ Degrees in the Arch whoſe Co-ſine is $\frac{y}{a}$, Radius $= 1$.

Ex. 13.

105. *Let AD be the Cycloid,* $AB = x$, $BD = y$, $AE = 2r$, $FC = u$, Arch $AC = v$; by the Nature of the Circle $2ry - yy = uu$, and $\dot{u} = \frac{r\dot{y} - y\dot{y}}{u}$; and $\dot{v} = \frac{r\dot{y}}{u}$: And

by the Nature of the Cycloid $x = v + u$, and $\dot{x} = \dot{v} + \dot{u} = \frac{r\dot{y}}{u} + \frac{r\dot{y} - y\dot{y}}{u} = \frac{2r\dot{y} - y\dot{y}}{\sqrt{2ry - yy}}$; whence $\dot{z} = y\dot{x} = \frac{2ry - yy}{\sqrt{2ry - yy}}\dot{y} = \dot{y}\sqrt{2ry - yy}$, but this is the Fluxion of the Area *AFC*, therefore z or *ABD* = *AFC*.

Ex. 14.

Let AD be the Quadratix, AB$=x$, *BD*$=y$, then (by the Proceſs in Ex. 11. Prob. VIII.) $\dot{x} = \frac{2y\dot{y}}{3a} + \frac{4y^3\dot{y}}{45a^3} + \frac{4y^5\dot{y}}{315a^5} + \&c.$ Whence $\dot{z} = y\dot{x} = \frac{2y^2\dot{y}}{3a} + \frac{4y^4\dot{y}}{45a^3} + \frac{4y^6\dot{y}}{315a^5}$ &c. whence $z = \frac{2y^3}{9a} + \frac{4y^5}{225a^3} + \frac{4y^7}{2205a^5} + \&c.$ 106.

Ex. 15.

Let AD be the Catenary, AB$=x$, *BD*$=y$ Curve *AD*$=v$, the Equation of the Curve is $vv = 2ay + yy$, whence $v\dot{v} = a\dot{y} + y\dot{y}$, and $v^2\dot{v}^2 = \overline{a+y}^2 \times \dot{y}^2$, or $\overline{a+y}^2 \times \dot{v}^2 - a^2\dot{v}^2 = \overline{a+y}^2 \times \overline{\dot{v}^2 - \dot{x}^2}$, whence $a\dot{v} = \overline{a+y} \times \dot{x}$; therefore $\dot{z} = y\dot{x} = a\dot{v} - a\dot{x}$, and $z = av - ax$. 107.

Ex. 16.

Let AD be ſuch a mechanical Curve, that the Ordinate FD = *Arch AC of the Parabola whoſe Equation is* $ry = uu$; putting *AB*$=x$, *BD* or *AF*$=y$, *FC*$=u$, Arch *AC*$=v$. Then $v = x$, and $\dot{x} = \dot{v} = (\sqrt{\dot{y}^2 + \dot{u}^2} =)$ $y^{-\frac{1}{2}}\dot{y}\sqrt{\frac{1}{4}r + y}$, by Prob. VIII. then $\dot{z} = y\dot{x} = y\dot{v} = y^{\frac{1}{2}}\dot{y}\sqrt{\frac{1}{4}r + y}$. But v = Fluent of $y^{-\frac{1}{2}}\dot{y}\sqrt{\frac{1}{4}r + y}$, there- 108.

FIG. fore by Form 11th, $z = \overline{\tfrac{1}{8}r + \tfrac{1}{2}y}\sqrt{\overline{\tfrac{1}{4}ry + yy}} - \dfrac{rv}{16}$.

Ex. 17.

109. *Let AD be a Circle, to find the Area ABD, extreamly near the Vertex.* Let Radius $CA = a$, $AB = x$, $BD = y$, then $\dot{z} = y\dot{x} = \dot{x}\sqrt{2ax - xx}$; but in A, $\sqrt{2ax - xx} = \sqrt{2ax}$; therefore (by Cor. 3. Prob. II.) $\dot{z} = y\dot{x} = \dot{x}\sqrt{2ax}$; and the Area $z = \tfrac{2}{3}x\sqrt{ax} = \tfrac{2}{3}xy$, the Area in the very Vertex of the Figure.

Cor. After the ſame Manner in all Curves of a finite Curvature, the Area of the Segment extreamly near the Vewtex is $\tfrac{2}{3}$ the Baſe into the Height.

Ex. 18.

Let the Equation of a Curve be $y = \dfrac{\overline{ax^r + bx^s + dx^t}|^m}{\overline{fx^v + gx^w}|^n}$, *to find the Area of an extreamly ſmall Part of the Figure,* when $x = c$. Here $\dot{z} = y\dot{x} = \dfrac{\overline{ac^r + bc^s + dc^t}|^m}{\overline{fc^v + gc^w}|^n} \times \dot{x}$ (by Cor. 3. Prop. II.) therefore $z = \dfrac{\overline{ac^r + bc^s + dc^t}|^m}{\overline{fc^v + gc^w}|^n} \times x'$. Where x' is a very ſmall Part of the Abſciſſa.

Cor. Hence the Area of any compound Curve may be nearly found, by finding after this Manner, all the Parts of the Area, belonging to the ſeveral ſmall Parts of the Abſciſa, and collecting them into one Sum.

Ex. 19.

110. *Let BDm be Archimedes' Spiral,* $AB = r$, Arch $AC = v$, $BD = y$, by the Nature of the Figure $rv = cy$,

whence $\dot{v} = \frac{c\dot{y}}{r}$, and by ſimilar Triangles $\dot{v} = \frac{r\dot{x}}{y}$ whence $\dot{x} = \frac{cy\dot{y}}{rr}$; therefore $\dot{z} = \frac{cyy\dot{y}}{2rr}$; and thence he Area *BED* or $z = \frac{cy^3}{6rr}$.

Ex. 20.

Let QDC be the reciprocal Spiral, Radius $AB = a$, Arch $AC = b$, Arch $AP = v$, $BD = y$; the Nature of the Curve gives $ab = vy$, whence $v\dot{y} + y\dot{v} = 0$, and $\dot{v} = \frac{-v\dot{y}}{y}$, and by ſimilar Triangles $\dot{v} = \frac{a\dot{x}}{y}$, whence $\dot{x} = \frac{-v\dot{y}}{a}$, therefore $\dot{z} = \frac{y\dot{x}}{2} = \frac{-vy\dot{y}}{2a} = \frac{-ab\dot{y}}{2a} = \frac{-b\dot{y}}{2}$, and $z = \frac{-by}{2}$; and when corrected, the Area *BCD* or $z = b \times \frac{BC - BD}{2}$. 111.

Ex. 21.

Let QDC be the *proportional Spiral*, whoſe Nature s $b\dot{y} = c\dot{x}$, or $\dot{x} = \frac{b\dot{y}}{c}$, therefore $\dot{z} = \frac{y\dot{x}}{2} = \frac{by\dot{y}}{2c}$: And z or the Area $BQD = \frac{by^2}{4c}$. 111.

Ex. 22.

Let BED be a Kind of Spiral, $BD = y$, Arch *AD* (whoſe Center is *B*) $= v$, and let its Nature be expreſs'd by this Equation $av^2 = y^3$, or $y = a^{-\frac{1}{3}}v^{\frac{2}{3}}$, and $\dot{y} = \frac{2}{3}a^{\frac{1}{3}}v^{-\frac{1}{3}}\dot{v}$, whence $\dot{z} = v\dot{y} = \frac{2}{3} \cdot a^{\frac{1}{3}}v^{\frac{2}{3}}\dot{v}$, and the Area $z = \frac{2}{5}a^{-\frac{1}{3}}v^{\frac{5}{3}} = \frac{2}{5}vy$ for the Area *ABED*. 112.

Ex. 23.

FIG.

Ex. 23.

Let the Equation of a Curve be $a^5x^2y^2 - x^9 = a^6y^3$, pu $y = \frac{xx}{z}$, and the Equation is transform'd into a^5z $- x^3z^3 = a^6$. Whence $x = \frac{\sqrt[3]{a^5z - a^6}}{z}$, $y = \frac{\overline{a^5z - a^6}|}{z^3}$ and $\dot{x} = \frac{\frac{1}{3}a^5z\dot{z} - a^5z\dot{z} + a^6\dot{z}}{zz \times \overline{a^5z - a^6}|^{\frac{2}{7}}}$ and $y\dot{x} =$ $\frac{\frac{1}{3}a^5z\dot{z} - a^5z\dot{z} + a^6\dot{z}}{z^5} = \frac{a^6\dot{z}}{z^5} - \frac{2a^5\dot{z}}{3z^4}$. And the Fluent $= \frac{-a^6}{4z^4} + \frac{2a^5}{9z^3} = \frac{-a^6y^4}{4x^8} +$ $\frac{2a^5y^3}{9x^6}$ for the Area.

Ex. 24.

237. *Let the Equation of the Curve be* $y^4 - 6a^2y^2 + 4x^2y$ $+ a^4 = 0$. By Reduction $x = \frac{\sqrt{-y^4 + 6a^2y^2 - a^4}}{2y}$ let $\frac{y^4 - 2a^2y^2 + a^4}{4yy} = vv$, then $\frac{yy - aa}{2y} = v$, and $\frac{aa - yy}{2y} = v$; and $aa - vv = aa - \frac{y^4 - 2a^2y^2 + a^4}{4yy}$ $= \frac{-y^4 + 6aayy - a^4}{4yy} = xx$. Therefore $x = \sqrt{aa - vv}$ From the two Values of v, we ſhall have $yy - aa =$ $\pm 2vy$, and $y = \pm v + \sqrt{aa + vv}$, and $\dot{y} = \pm \dot{v} +$ $\frac{v\dot{v}}{\sqrt{aa + vv}}$, then the Fluxion of the Area $x\dot{y} = \pm \dot{v}$ $\sqrt{aa - vv} + \frac{v\dot{v}\sqrt{aa - vv}}{\sqrt{aa + vv}} = \pm \frac{aa\dot{v} - v^2\dot{v}}{\sqrt{aa - vv}} +$

$aa\dot{v}$

FIG.

$\frac{aav\dot{v} - v^3\dot{v}}{\sqrt{a^4 - v^4}}$; whoſe Fluent is had from Form 10, *c.* of the Table.

Ex. 25.

Let the Equation of the Curve be $x^3 + y^3 = axy$. 238.

ut $y = \frac{axx}{vv}$, and the Equation is transformed to $^6 + x^3a^3 = a^2v^4$. Then $x^3 = \frac{a^2v^4 - v^6}{a^3}$, and $3x^2\dot{x}$ $= \frac{4a^2v^3\dot{v} - 6v^5\dot{v}}{a^3}$, and the Fluxion of the Area $y\dot{x}$ $= \frac{axx}{vv} \times \frac{4a^2v^3\dot{v} - 6v^5\dot{v}}{3x^2a^3} = \frac{4a^2v\dot{v} - 6v^3\dot{v}}{3aa}$, whoſe luent is $\frac{2a^2v^2 - \frac{3}{2}v^4}{3aa} = \frac{2vv}{3} - \frac{v^4}{2aa} = \frac{2ax^2}{3y} - \frac{x^4}{2yy}$.

Ex. 26.

To find the Quadrature of the Curve defined by the quation $ax^\lambda + bx^\mu y^\nu + cy^\tau = 0$.

Put $y\dot{x} = u\dot{z}$, and $z = \frac{x^s}{s}$. Then $y = \frac{u\dot{z}}{\dot{x}} =$ $x^{s-1} = us^{\frac{s-1}{s}} z^{\frac{s-1}{s}}$, becauſe $x = \overline{sz}^{\frac{1}{s}}$. Therefore he Equation is transform'd into this $a^{\frac{\lambda}{s}} z^{\frac{\lambda}{s}} + bu^\nu \times$ $^{\frac{\mu+\overline{s-1}\cdot\nu}{s}} \times z^{\frac{\mu+\overline{s-1}\cdot\nu}{s}} + cu^\tau s^{\frac{\overline{s-1}\cdot\tau}{s}} \times z^{\frac{s-1}{s}\tau} = 0$. Diide the Equation by $s^{\frac{\mu+\overline{s-1}\cdot\nu}{s}} \times z^{\frac{\mu+\overline{s-1}\cdot\nu}{s}}$, and we hall have $as^{\frac{\lambda-\mu-\overline{s-1}\cdot\nu}{s}} \times z^{\frac{\lambda-\mu-\overline{s-1}\cdot\nu}{s}} + bu^\nu + cu^\tau \times$

s

$s^{\frac{\overline{s-1}.\overline{\tau-\nu}-\mu}{s}} \times z^{\frac{\overline{s-1}.\overline{\tau-\nu}-\mu}{s}} = 0$. Now to make z vaniſh out of the laſt Term, put its Index $\frac{\overline{s-1}.\overline{\tau-\nu}-\mu}{s} = 0$, then $s = \frac{\mu}{\tau-\nu} + 1$; and $\frac{\lambda-\mu-\overline{s-1}.\nu}{s} = \frac{\lambda\tau-\mu\tau-\lambda\nu}{\tau-\nu+\mu}$ $= \pi$, by Subſtitution. Then the Equation comes to this $as^{\pi}z^{\pi} + bu^{\nu} + cu^{\tau} = 0$, and conſequently $sz =$ $\overline{-\frac{b}{a}u^{\nu} - \frac{c}{a}u^{\tau}}|^{\frac{1}{\pi}}$. Therefore $F:u\dot{z} = uz - F:$ $z\dot{u} = uz - F: \frac{\dot{u}}{s} \times \overline{-\frac{b}{a}u^{\nu} - \frac{c}{a}u^{\tau}}|^{\frac{1}{\pi}}$. But $u\dot{z} =$ $y\dot{x}$; alſo $z = \frac{x^{s}}{s}$, and $uz = \frac{ux^{s}}{s} = \frac{x}{s} \times ux^{s-1} =$ $\frac{xy}{s}$. Therefore

$$F:y\dot{x} = \frac{xy}{s} - F: \frac{\dot{u}}{s} \times \overline{\frac{-b}{a}u^{\nu} - \frac{c}{a}\dot{u}^{\tau}}|^{\frac{1}{\pi}},$$ the

Area. Hence

1. For Example, let $x^3-axy+y^3=0$, then $F:y\dot{x}$ $= \frac{2xy}{3} - F: \frac{2}{3}\dot{u} \times \overline{au-u^3}|$.

2. Let $x^5-x^3y^2+g^2y^3=0$, then $F:y\dot{x} = \frac{xy}{4} - F:$ $\frac{\dot{u}}{4\times\overline{uu-ggu^3}}$.

SCHOLIUM.

It may not be improper, in this Place, to inſert (from the Tranſactions) a general Method for determining the Quadratures of Curves, with its Inveſtigation.

Let the Equation of a Curve be $y^m + ax^n + by^p x^q + cy^r x^s + dy^t x^u$ &c. $= 0$. And its Area $Ayx + By^l x^f + Cy^g x^h + Dy^i x^k +$ &c. $=$ Fluent of $y\dot{x}$. Put theſe two Equations into Fluxions, and make the two Values of $\dot{y}$ equal to each other (expunging y^{m-1} by the Equation of the Curve) and we have

$$\frac{anx^{n-1} + bqy^p x^{q-1} + csy^r x^{s-1} + \text{\&c.}}{-max^n y^{-1} \substack{+p\\-m} by^{p-1} x^q \substack{+r\\-m} cy^{r-1} x^s \text{ \&c.}} = \frac{\substack{A\\-1} y + fBy^l x^{f-1} + hCy^g x^{h-1} + \text{\&c.}}{Ax + lBy^{l-1} x^f + gCy^{g-1} x^h \text{ \&c.}};$$

and multiplying the Numerators and Denominators alternately, there ariſes,

$$\left.\begin{array}{lll} anAx^n + bqAy^p x^q & + csAy^r x^s \\ \quad + anlBy^{l-1} x^{f+n-1} & + lbqBy^{p+l-1} x^{f+q-1} \\ & + ganCy^{g-1} x^{h+n-1} \end{array}\right\} \text{\&c.} =$$

$$\left.\begin{array}{lll} \overline{1-A}.max^n + \overline{1-A} \times \substack{m\\-p} by^p x^q & + \overline{1-A} \times \substack{+m\\-r} cy^r x^s \\ \quad - mafBy^{l-1} x^{f+n-1} & \substack{+p\\-m} bfBy^{p+l-1} x^{f+q-1} \\ & + mahCy^{g-1} x^{h+n-1} \end{array}\right\} \text{\&c.}$$

Now for determining the Indices, compare the homologous Terms. Thus $p = l - 1$, whence $l = p + 1$. Again $r = p + l - 1 = g - 1$, and thence $r = 2p$, and $g = 2p + 1$. After the ſame Manner $t = 3p$, $i = 3p + 1$. Put $e + n = q$, and then $f = e + 1$, $s = 2e + n$, $h = 2e + 1$, $u = 3e + n$, $k = 3e + 1$, &c. Therefore the Equation of the Curve will be in this Form, $y^m + ax^n + by^p x^{e+n} + cy^{2p} x^{2e+n} + dy^{3p} x^{3e+n}$ &c. $= 0$, and its Quadrature $Ayx + By^{p+1} x^{e+1} + Cy^{2p+1} x^{2e+1} + Dy^{3p+1} x^{3e+1}$ &c. $=$ Fluent of $y\dot{x}$.

Next for determining the Coefficients A, B, C, *&c.* Put the Coefficients of the homologous Powers of x equal to each other, thus $anA = \overline{1 - A} \times ma$, whence $A = \frac{m}{m+n}$. After the ſame Manner $B =$

$$\left.\begin{array}{l} m - p \\ + \overline{p - m - e - n} \times A \end{array}\right\} \times \frac{b}{am \times \overline{e+1} + an \times \overline{p+1}},\ C =$$

$$\left.\begin{array}{l} \overline{m - 2p} \times c \\ + \overline{2p - 2e - m - n} \times cA \\ : + \overline{m-p}.\overline{e+1} + .\overline{e+n}.\overline{p+1} \times : -bB \end{array}\right\} \times \frac{1}{a \times : m \times \overline{2e+1} + n \times \overline{2p+1} :}$$

&c.

Hence therefore if $y^m + ax^n + by^p x^{e+n} + cy^{2p} x^{2e+n} + dy^{3p} x^{3e+n}$ *&c.* $= 0$, *then the Fluent of* $y\dot{x}$, *or the Area of that Curve is,*

$$+ \frac{m}{m+n} yx.$$

$$\left.\begin{array}{l} + \overline{m - p} \times b \\ + \overline{p - m - e - n} \times Ab \end{array}\right\} \times \frac{y^{p+1} x^{e+1}}{am \times \overline{e+1} + an \times \overline{p+1}}$$

$$\left.\begin{array}{l} + \overline{m - 2p} \times c \\ + \overline{2p - 2e - m - n} \times cA \\ + : \overline{m - p} \times \overline{e+1} + \overline{e+n} \times \overline{p+1} : \times -bB \end{array}\right\} \times \frac{y^{2p+1} x^{2e+1}}{am \times \overline{2e+1} + an \times \overline{2p+1}}$$

$$\left.\begin{array}{l} + \overline{m - 3p} \times d \\ + \overline{3p - 3e - m - n} \times dA \\ + : \overline{m-2p} \times \overline{e+1} + \overline{2e+n} \times \overline{p+1} : \times -cB \\ + : \overline{m-p} \times \overline{2e+1} + \overline{e+n} \times \overline{2p+1} : \times -bC \end{array}\right\} \times \frac{y^{3p+1} x^{3e+1}}{am \times \overline{3e+1} + an \times \overline{3p+1}}$$

$$\left.\begin{array}{l} + \overline{m - 4p} \times e \\ + : \overline{4p - 4e - m - n} : \times eA \\ + : \overline{m-3p} \times \overline{e+1} + \overline{3e+n} \times \overline{p+1} : \times -dB \\ + : \overline{m-2p} \times \overline{2e+1} + \overline{2e+n} \times \overline{2p+1} : \times -cC \\ + : \overline{m-p} \times \overline{3e+1} + \overline{e+n} \times \overline{3p+1} : \times -bD \end{array}\right\} \times \frac{y^{4p+1} x^{4e+1}}{ma \times \overline{4e+1} + na \times \overline{4p+1}}$$

&c.

1. Here note, A, B, C, D *&c.* repreſent the firſt, ſecond, third, fourth, *&c.* Terms, only leaving out the Powers of x and y.

2. This Series ſometimes gives the Quadrature of a Curve in finite Terms; particularly in Trinomials (that is whoſe Equation conſiſts of three Terms), when $\frac{e - p + m + n}{- em - pn}$ is a poſitive whole Number; to which add 1, and you have the Number of Terms in the Series.

3. But in Curves of more Terms, there are ſeveral Conditions requiſite to their exact Quadrability, which it is needleſs to enumerate, becauſe ſuch Curves ſeldom admit of an exact Quadrature. It is ſufficient to obſerve, that if $N =$ Number of Terms in the Equation of the Curve, they will ſometimes admit of ſuch a Quadrature when $\frac{Ne - 2e + 2p - Np + m + n}{- em - pn} + 1$ is a poſitive whole Number, but never elſe; which Number will then ſhew the Number of Terms in the Series, that conſtitutes the Area.

4. When the Quadrature of a Curve is required by this Series; reduce the Equation of the Curve to the preceding Form, and comparing the homologous Terms, the Exponents and Coefficients will be eaſily determined; which muſt be ſubſtituted into the foregoing general Series as uſual. And when any particular Terms are wanting in the given Equation, then the reſpective Coefficients will be 0, and thoſe Terms of the Series wherein they are found will vaniſh.

5. And we muſt firſt of all enquire whether it will admit of an exact or geometrical Quadrature, and if it will not admit of it in one Form, it may in another. To this Purpoſe we muſt divide the whole Equation by ſome Powers of x and y; ſo that in the reſulting Equation, there

may always be one Term without x and another without y; for this Condition is abſolutely neceſſary to the Equation; and thus you will have a new Form: And this we muſt do as oft as poſſible. Now the Number of different Forms any Equation will admit of is $NN-N$, putting $N=$Number of Terms in the Equation.

6. If it admit of ſuch Quadrature in none of theſe Forms, try to find the Complement of the Area, by writing x for y and y for x in the Equation of the Curve; and then proceeding with the new Equation in all Reſpects as before.

EXAMPLE I.

Let $y^3+x^3-byx=0$. Here $m=3$, $n=3$, $p=1$, $e=-2$, $a=1$, $b=-b$, and $\frac{e-p+m+n}{-em-pn.}=1$; whence the Curve is geometrically quadrable, and the Area $=\frac{1}{2}xy-\frac{y^2b}{bx}$.

Ex. 2.

Suppoſe $y^{\frac{5}{3}}-ax^{\frac{1}{4}}+ry^7x^{\frac{3}{2}}=0$. Here $m=\frac{5}{3}$, $n=\frac{1}{4}$, $a=-a$. $b=r$, $p=7$, $e=\frac{5}{4}$: then $\frac{e-p+m+n}{-em-pn}=1$; and the Area $=\frac{20}{23}xy+\frac{8r}{23a}y^8x^{\frac{9}{4}}$.

Ex. 3.

Let $y^2-\frac{4g^2}{b}x^6+\frac{g}{b}y^2x^4=0$; here the Curve is not quadrable in this Form, therefore divide by y^2, and then $1+\frac{g}{b}x^4-\frac{4gg}{b}y^{-2}x^6=0$, where $m=0$,

$n=4$, $p=-2$, $e=2$, $a=\frac{g}{b}$, $b=-\frac{4gg}{b}$, and $\frac{e-p+m+n}{-em-pn}=1$; whence the Area $=\frac{2gx^3}{y}$.

Ex. 4.

Let $y^2-ax^{-1}-2ay^3x-ay^6x^3=0$; here $m=2$, $n=-1$, $a=-a$, $b=-2a$, $c=-a$, $p=3$, $e=2$, whence $\frac{2e-2p+m+n}{-em-pn}=1$; and the Area of the Figure is $=2x\dot{y}-\frac{2}{5}x^3y^4$: For all the following Terms of the Series will be nothing.

PROB. XI.

To find any Number of Curves that may be ſquared.

Let the Abſciſſa $AB=x$, *Ordinate* $BD=y$, *Area* $ABD=z$. *Aſſume any Equation between* x *and* z, *this will determine the Area: From that Equation get* $\dot{z}$, *and ſubſtitute its Value in the Equation* $\dot{z}=y\dot{x}$, *and this will give the Nature of the Curve.*

FIG. 113.

EXAMPLE I.

Let $x^2=z$, whence $2x\dot{x}=\dot{z}=y\dot{x}$, whence $y=2x$, and the Figure is a Triangle.

Ex. 2.

Let $ax^3=z^2$, and $\dot{z}=\frac{3}{2}a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}=y\dot{x}$, whence $\frac{3}{2}\sqrt{ax}=y$, an Equation to a Parabola.

Likewiſe if $x^3=az$, then $\frac{3x^2\dot{x}}{a}=\dot{z}=y\dot{x}$; whence $3x^2=ay$, an Equation, again, to the Parabola.

Ex. 3.

Ex. 3.

Let $\frac{a^6}{xx} = z^2$, whence $-\frac{a^3}{xx}\dot{x} = \dot{z} = y\dot{x}$; therefore $-\frac{a^3}{xx} = y$, whence y being negative, lies on the other Side of *AB*.

Ex. 4.

Let $ccaa + ccxx = zz$, or $z = c\sqrt{aa+xx}$, then $\dot{z} = \frac{cx\dot{x}}{\sqrt{aa+xx}} = y\dot{x}$; therefore $y = \frac{cx}{\sqrt{aa+xx}}$.

Ex. 5.

If $\frac{\overline{aa+xx}|^{\frac{3}{2}}}{b} = z$, then $\dot{z} = \frac{3x\dot{x}}{b}\sqrt{aa+xx} = y\dot{x}$, and therefore $y = \frac{3x}{b}\sqrt{aa+xx}$.

Ex. 6.

Aſſume $b - 3xz + \frac{2}{3}z = z^2$, then $\frac{2}{3}\dot{z} - 3x\dot{z} - 3z\dot{x} = 2z\dot{z}$, and $\dot{z} = \frac{3z\dot{x}}{\frac{2}{3} - 3x - 2z} = y\dot{x}$; therefore $y = \frac{3z}{\frac{2}{3} - 3x - 2z}$, out of which z may be expunged by Help of the aſſumed Equation.

Ex. 7.

Let $\overline{e+fx^n}|^{m} = z$, then $\dot{z} = mnfx^{n-1}\dot{x} \times \overline{e+fx^n}|^{m-1} = y\dot{x}$; whence $y = mnfx^{n-1} \times \overline{e+fx^n}|^{m-1}$.

PROB.

PROB. XII.

Any Curve ABD being given; to find any Number of Curves ACE whoſe Areas ſhall have any aſſigned Relation to the Area of the given Curve. FIG.

Let *ADF* be any Curve referred to an Axis, or 114.
ADF any mechanical Curve, whoſe Complement is 115.
ABD; or *ADH* any Spiral deſcribed by the Arch 116.
BD, whoſe Center is *A*. *ACE* the Curve required, 117.
whoſe Relation to the firſt is given.

Put $AB=x$, $BD=y$, Area $ABD=z$.
$AC=v$, $CE=u$, Area $ACE=w$.

Aſſume any two Equations, one of which may contain he Relation of the Areas z, w; *the other the Relation f the Abſciſſa or Ordinate of the given Curve AD x or y) to the Abſciſſa or Ordinate (v or u) of the ther Curve AE. By Help of theſe two Equations, and his third* $y\dot{x}=\dot{z}$, *expunge all Quantities as far as poſſible, xcept v and u, out of the Equation* $u\dot{v}=\dot{w}$, *and you vill get an Equation for the Nature of the Curve ACE: And the ſecond aſſumed Equation will determine the Quantities of the Abſciſſas or Ordinates of the two Curves o have the required Relation.*

Ex. 1. Let *AD* be a Circle, whoſe Equation is 114.
$x-xx=yy$. Aſſume $ax=vv$, and $z=w$, then $\dot{x}=$ 117.
$\frac{v\dot{v}}{a}$; whence $u\dot{v}=\dot{w}=\dot{z}=y\dot{x}=\frac{2v\dot{v}}{a}\sqrt{ax-xx}=$
$\frac{v\dot{v}}{a}\sqrt{vv-\frac{v^4}{aa}}$; therefore $u=\frac{2vv}{aa}\sqrt{aa-vv}$, the
quation of the Curve whoſe Area is equal to that
f the Circle, when $ax=vv$.

Ex. 2,

FIG. 114. 117. Ex. 2. Let $ax - xx = yy$ as before, and $ax + z = v$, and $z = w$. Then $u\dot v = \dot w = \dot z = y\dot x = \frac{\dot v - \dot z}{a} y =$ $\frac{\dot v - u\dot v}{a} y$: Whence $u = \frac{y - uy}{a}$, or $u = \frac{y}{a+y}$, an Equation to a mechanical Curve.

Ex. 3. Again let $ax - xx = yy$, $cx + z = w$, and ax $= vv$. Then $u\dot v = \dot w = c\dot x + \dot z = c\dot x + y\dot x =$ $\frac{2cv\dot v}{a} + \frac{2v\dot v}{a}\sqrt{v^2 - \frac{v^4}{aa}}$. Whence $u = \frac{2vv}{aa} \times$ $\sqrt{aa - vv} + \frac{2cv}{a}$.

Ex. 4. Let $ax - xx = yy$ as before, and aſſume $z -$ $\frac{2y^3}{3a} = w$, and $x = v$. Then $u\dot v = \dot w = \dot z - \frac{2y^2\dot y}{a}$ $= y\dot x - \frac{2y^2\dot y}{a} = \frac{2x\dot x}{a} \times \sqrt{ax - xx} = \frac{2v\dot v}{a}$ $\sqrt{av - vv}$; hence $u = \frac{2v}{a}\sqrt{av - vv}$.

Ex. 5. Again let $ax - xx = yy$, and $z^2 = w$, $x = v^2$. Then $u\dot v = \dot w = 2z\dot z = 2zy\dot x = 4zyv\dot v = 4zv\dot v\sqrt{ax - xx}$ $= 4zv\dot v\sqrt{av^2 - v^4} = 4zv^2\dot v\sqrt{a - v^2}$: Hence $u =$ $4zvv\sqrt{a - vv}$, an Equation to a mechanical Curve.

98. 117. Ex. 6. Let $cc + xx = yy$ an Equation to an Hyperbola; and aſſume $z = w$, $xx = cv$. Then $u\dot v = \dot w =$ $\dot z = y\dot x = \frac{c\dot v}{2\sqrt{cv}} \times \sqrt{cc + cv} = \frac{c\dot v}{2v}\sqrt{cv + vv}$. Hence

$$u = \frac{c}{2v}\sqrt{cv + vv}.$$

Ex. 7. Let $cc + xx = yy$ as before, and $xy - z = w$, $xx = cu$. Then $u\dot v = \dot w = x\dot y + y\dot x - \dot z = x\dot y =$ $\frac{x\dot x}{\sqrt{cc + xx}} = \frac{cv\dot v}{2\sqrt{cv + vv}}$. Hence $u = \frac{cv}{2\sqrt{cv + vv}}$.

Ex. 8. Let $y = \frac{xx}{\sqrt{ax-xx}}$ an Equation to the FIG. 103.
Ciſſoid; and $\frac{2x}{3}\sqrt{ax-xx} + \frac{1}{3}z = w$, and $x = v$. 117.
Then $u\dot{v} + \dot{w} = \frac{2}{3}\dot{x} \times \sqrt{ax-xx} + \frac{2ax\dot{x} - 4x^2\dot{x}}{6\sqrt{ax-xx}} + \frac{1}{3}\dot{z}$
$= \frac{ax-xx}{\sqrt{ax-xx}}\dot{x} = \dot{x}\sqrt{ax-xx} = \dot{v}\sqrt{av-vv}$.
Hence $u = \sqrt{av-vv}$, an Equation to the Circle.

Ex. 9. Let $y = dx^r \times \overline{e+fx^n}|^m$; aſſume $z = w$, $v =$ 114.
$\overline{+fx^n}|^m$, then will $x = \overline{\frac{v^{\frac{1}{m}} - e}{f}}|^{\frac{1}{n}}$, and $\dot{x} = \frac{v^{\frac{1}{m}-1}\dot{v}}{mnf^{\frac{1}{n}}}$ 117.
$\times\, \overline{v^{\frac{1}{m}} - e}|^{\frac{1}{n}-1}$, $y = dv \times \overline{\frac{v^{\frac{1}{n}} - e}{f}}|^{\frac{r}{n}}$; whence $u\dot{v} =$
$\dot{v} = \dot{z} = y\dot{x} = \frac{dv^{\frac{1}{m}}\dot{v}}{mnf^{\frac{r+1}{n}}} \times \overline{v^{\frac{1}{m}} - e}|^{\frac{r+1}{n}-1}$: there-
ore $u = \frac{dv^{\frac{1}{m}}}{mnf^{\frac{r+1}{n}}} \times \overline{v^{\frac{1}{m}} - e}|^{\frac{r+1}{n}-1}$.

Ex. 10. Let *BD* or $y =$ Arch *AR* of the Parabo- 118.
a *AR*, whoſe Equation is $ax = ss$, putting $AR = p$, 117.
$BR = s$, Tangent $TR = t$; then $\dot{s} = \frac{s\dot{p}}{t}$. Now aſ-
ume $z = w$, $y = v$, then will $u\dot{v} = \dot{w} = \dot{z} = y\dot{x} =$
$\frac{vss}{a} = \frac{2vss\dot{p}}{at} = \frac{2vx}{t}\dot{p} = \frac{2vx}{t}\dot{y} = \frac{2vx}{t}\dot{v}$;

FIG. conſequently $u = \frac{2vx}{t}$, a mechanical Curve.

115. 117. Ex. 11. Let *AD* be a Cycloid, Diameter of the generating Circle $= a$, $FG = s$. Aſſume $y = v$, $z = w$, then by the Property of the Cycloid $y\dot{x} = s\dot{y}$; whence $u\dot{v} = \dot{w} = \dot{z} = y\dot{x} = s\dot{y} = s\dot{v}$; and therefore $u = s$; conſequently *AE* is a Circle the ſame with *AG*.

Ex. 12. Let *AD* be a Figure of Arches, where Arch $AG = AB = x$. Aſſume $y = v$, $z = w$. And by the Property of the Circle $\dot{x} = \frac{a\dot{y}}{2\sqrt{ay - yy}}$.

Then $u\dot{v} = \dot{w} = \dot{z} = y\dot{x} = \frac{av\dot{y}}{2\sqrt{ay - yy}} = \frac{av\dot{v}}{2\sqrt{av - vv}}$.

Hence $u = \frac{av}{2\sqrt{av - vv}}$.

116. 117. Ex. 13. Let *AD* be the Spiral of *Archimedes*, whoſe Equation is $by = xx$; and aſſume $x = v$, and $z = w$; then $u\dot{v} = \dot{w} = \dot{z} = y\dot{x} = y\dot{v} = \frac{xx}{b}\dot{v} = \frac{vv}{b}\dot{v}$; whence $bu = vv$, and *AE* is a Parabola, convex towards *AC*.

PROB. XIII.

To find the Surface of a Solid.

As the Fluxion of any Space is equal to the deſcribing Line drawn into the Fluxion of the Axis; ſo the Fluxion of the Surface of a Solid (generated by a Line revolving about an Axis) is equal to the Periphery of that Circle drawn into the Fluxion of the Line generating that Surface. Therefore

Let the Abſciſa $AP=x$, *Ordinate* $PM=y$, *Curve* FIG.
AM (or BM) $=z$, $c=3.1416\times 2=$ *the Circumference* 119.
of the Circle whoſe Radius is 1; *then to find the Surface* 120.
s generated by the Curve AM revolving about the Axis 121.
AP; *by the Equation of the Curve expunge one of the ndetermined Quantities and its Fluxion out of the Equation* $\dot{s}=cy\dot{z}$, or $cy\sqrt{\dot{x}^2+\dot{y}^2}$; *and find the Fluent.*

EXAMPLE 1.

Let ABD be a right Cone, $AB=a$, $BC=b$, then 123.
$y=\frac{bz}{a}$, therefore $\dot{s}=cy\dot{z}=\frac{cbz\dot{z}}{a}$, and $s=\frac{cbz^2}{2a}$
$=\frac{czy}{2}$. And the Surface of the whole Cone *ABD*
$=\frac{c}{2}\times AB\times BC$.

Ex. 2.

Let AM be a ſpherical Surface, Radius $=a$, by the 119.
Property of the Circle $a\dot{x}=y\dot{z}$; therefore $\dot{s}=cy\dot{z}$
$=ca\dot{x}$, whence $s=cax$.

Ex. 3.

Let AM be a parabolic Surface, $ay=xx$, then $\dot{y}=$ 120.
$\frac{2x\dot{x}}{a}$, whence $\dot{s}=cy\sqrt{\dot{x}^2+\dot{y}^2}=\frac{cx^2\dot{x}}{aa}\sqrt{aa+4xx}$:
and the Fluent (corrected) is $s=\frac{caax+8cx^3}{32aa}\sqrt{aa+4xx}$
$-\frac{ca^2}{64}\times 2.302585\ \text{Log.}\ \frac{2x+\sqrt{aa+4xx}}{a}$, found
by Forms the 9th, 11th, and 13th.

Ex. 4.

Let $ax^m=y^{m+1}$ *be an Equation to infinite Parabolas.* 119.
By the Proceſs in Ex. 4. Prob. VIII. $\dot{z}=\dot{y}\sqrt{1+by^{\frac{2}{m}}}$;

FIG. 119. whence $\dot{s} = cy\dot{y}\sqrt{1+by^{\frac{2}{m}}} = cy^{\frac{1}{m}+1}\dot{y}\sqrt{b+y^{-\frac{2}{m}}}$.

Therefore the Fluent of $\dot{s}$ or $cy\dot{y}\sqrt{1+by^{\frac{2}{m}}}$ will be had by Form the 15th, when m is the half of any negative odd Number: And the Fluent of $cy^{\frac{1}{m}+1}\dot{y}\sqrt{b+y^{-\frac{2}{m}}}$ will be had alſo by Form the 15th, when m is any poſitive whole Number. And likewiſe the Fluent of $cy\dot{y}\sqrt{1+by^{\frac{2}{m}}}$ will be had by Form the 11th when m is the half of an odd Number; firſt finding the Fluent of $cy^{\frac{1}{m}-1}\dot{y}\sqrt{1+by^{\frac{2}{m}}}$, by Form the 9th and 13th. In other Caſes, Form the 15th or 16th will give the Fluent by infinite Series.

COR. If $m=1$, then $s = \frac{c}{12a} \times \overline{aa+4yy}^{\frac{3}{2}} - \frac{caa}{12}$.

EX. 5.

121. *In the elliptic Surface BM,* whoſe Center is A, let $AD = a$, $AB = b$, $AP = x$, $PM = y$; then $y = \sqrt{bb - \frac{bb}{aa}xx}$, and $\dot{y} = \frac{-bbx\dot{x}}{a\sqrt{aabb-bbxx}}$, and $\dot{z} = \sqrt{\dot{x}^2+\dot{y}^2} = \frac{b\dot{x}}{aay}\sqrt{a^4-aaxx+bbxx} = \frac{b\dot{x}}{aay}\sqrt{a^4\mp ddxx}$, putting $aa \backsim bb = dd$. Therefore $\dot{s} = cy\dot{z} = \frac{cb\dot{x}}{aa}\sqrt{a^4\mp ddxx}$. Let $Q = ,017453 \times$ Degrees in the Arch whoſe Sine is $\frac{dx}{aa}$, when a is greater than b. Or $Q = 2.30258$ Log. $\frac{dx+\sqrt{a^4+ddxx}}{aa}$, when a is leſs than b. And we ſhall have (by Form 10th and 13th,

or by Form the 9th and 13th) $s = \frac{cbaa}{2d}\mathcal{Q} + \frac{cbx}{2aa}\sqrt{a^4+\overline{bb-aa}.x^2}$, for the Surface *BM* revolving round *AP*.

Ex. 6.

In the Hyperboloid BM, deſcribed by revolving about 124.
AP, let the Semi-conjugate $= b$, Semi-tranſvers $AB=a$, $AP=x$, $PM=y$; then $y=\sqrt{\frac{bbxx}{aa}-bb}$, $\dot{y}=\frac{bx\dot{x}}{a\sqrt{xx-aa}}$, and $\sqrt{\dot{x}^2+\dot{y}^2}=\frac{b\dot{x}}{aay}\sqrt{ddxx-a^4}$, putting $dd=aa+bb$. Therefore $\dot{s}=cy\sqrt{\dot{x}^2+\dot{y}^2}=\frac{cb\dot{x}}{aa}\sqrt{ddxx-a^4}$; whence (by Form the 9th and 13th) $s=\frac{cbx}{2aa}\sqrt{ddxx-a^4}-\frac{cbaa}{2d}\times 2.30258$ Log. $dx+\sqrt{ddxx-a^4}$: And the Fluent corrected, $s=\frac{cbx}{2aa}\sqrt{ddxx-a^4}-\frac{cbb}{2}+\frac{cbaa}{2d}\times 2.30258$ Log. $\frac{da+ba}{dx+\sqrt{ddx^2-a^4}}$.

Ex. 7.

To find the Surface deſcribed by the Hyperbola revolving 124.
round the Conjugate AC. Here $x=\frac{a}{b}\sqrt{bb+yy}$, and $\dot{x}=\frac{ay\dot{y}}{b\sqrt{bb+yy}}$, and $\sqrt{\dot{x}^2+\dot{y}^2}=\dot{y}\sqrt{\frac{bb+\frac{dd}{bb}yy}{bb+yy}}$. The Circumference deſcribed by *M* is *cx*, whence $\dot{s}=cx\dot{z}=\frac{ca}{b}\dot{y}\sqrt{bb+\frac{dd}{bb}yy}$. And (by Form 9 and 13)

13) $s = \frac{cay}{bb}\sqrt{bb + \frac{dd}{bb}yy} + \frac{2.302585cabb}{2d} \times$ Log. $\frac{dy}{2b} + \sqrt{1 + \frac{ddyy}{b^4}}$.

Ex. 8.

122. *In the right angled Hyperbola CM*, let $AP = x$, $PM = y$, and $aa = xy$, and $\dot{y} = \frac{-y\dot{x}}{x} = \frac{-aa\dot{x}}{xx}$, whence $\sqrt{\dot{x}^2 + \dot{y}^2} = \frac{\dot{x}}{xx}\sqrt{a^4 + x^4}$, and therefore $\dot{s} = cy\sqrt{\dot{x}^2 + \dot{y}^2} = \frac{caa\dot{x}}{x^3}\sqrt{a^4 + x^4} = \frac{ca^6x^{-5}\dot{x}}{\sqrt{1 + a^4x^{-4}}} + \frac{caax\dot{x}}{\sqrt{a^4 + x^4}}$. Whence (by Form 3 and 9) $s = \frac{caa}{2} \times 2.30258 \times \text{Log.}\, x^2 + \sqrt{a^4 + x^4} : -\frac{caa}{2xx}\sqrt{a^4 + x^4}$. But in C, let $x = a$, and $s = 0$, therefore the Fluent corrected is $s = \frac{caa}{\sqrt{2}} - \frac{caa}{2xx}\sqrt{a^4 + x^4} + \frac{caa}{2} \times 2.30258\, \text{Log.}\, \frac{x^2 + \sqrt{a^4 + x^4}}{aa \times \overline{1 + \sqrt{2}}}$, for the Surface deſcribed by *CM* about the Aſſymptote *AP*.

Ex. 9.

125. *Let the Ellipſis Mm revolve round the Line AP perpendicular to the Axis CA.* Let $CA = d$, $RM = Qm = z$, then $\dot{s} = c\dot{z} \times PM + c\dot{z} \times Pm = 2cd\dot{z}$. Whence $s = 2cdz$. And the whole Surface deſcribed by $RDQ = cd \times$ Semi-periphery *RDQ*.

Ex. 10.

126. *To find the Surface of the Ungula of a Cylinder cut off by a Plane FQ.* Let *F* be the Vertex, *ADZ* the

Section made by a Circle perpendicular to the Axis; FIG. 126. *FD*, *NI* Perpendiculars on the Plane of the Circle *ADZ*; *DC*, *IP* Perpendiculars on *AZ*.

Put $CD=d$, $DF=a$, Radius of the Cylinder $=r$, $DI=z$, $CP=x$, then $r-d+PI=$ Coſine of $z=y$, or $PI=y+d-r$; and by the Nature of the Circle $r\dot{x}=y\dot{z}$, and by ſimilar Triangles $NI=\frac{a}{d}\times\overline{y+d-r}$; but $\dot{s}=NI\times\dot{z}=\frac{a\dot{z}}{d}\times\overline{y+d-r}=\frac{ar\dot{x}}{d}+\frac{a\dot{z}}{d}\times\overline{d-r}$: Whence $s=\frac{arx}{d}+\frac{az}{d}\times\overline{d-r}$; and the whole Surface $ZFADZ=a\times ADZ-\frac{ar}{d}\times\overline{\text{Arch } ZDA-\text{Cord } ZA}$.

Ex. 11.

127. *Let the Ciſſoid NM revolve round the Aſſymptote AP*; let $AN=a$, $AP=x$, $PM=y$, then is $x=\frac{\overline{a-y}|^{2}}{\sqrt{ay-yy}}=\frac{\overline{a-y}|^{\frac{3}{2}}}{\sqrt{y}}$; whence $\dot{x}=\frac{-a\dot{y}-2y\dot{y}}{2yy}\sqrt{ay-yy}$; therefore $\dot{z}=\frac{a\dot{y}}{2y}\sqrt{\frac{a+3y}{y}}$; conſequently $\dot{s}=cy\dot{z}=\frac{ca\dot{y}}{2}\sqrt{\frac{a+3y}{y}}=\frac{cay^{-\frac{1}{2}}\dot{y}}{2}\sqrt{a+3y}$: From whence (by Forms the 9th and 13th) $s=\frac{1}{2}ca\sqrt{ay+3yy}+\frac{caa}{4}\times 2.3025\,\text{Log.}\,a+6y+2\sqrt{3ay+9y^{2}}$: and the Fluent being duly corrected, $s=\frac{1}{2}ca\sqrt{ay+3yy}-caa+\frac{2.302585caa}{4}\times\text{Log.}\,\frac{a+6y+2\sqrt{3ay+9yy}}{a\times\overline{7+4\sqrt{3}}}$.

Ex. 12.

Ex. 12.

FIG. 128. *Let DM be the Log. Curve; to find the Surface it deſcribes by revolving round the Aſſymptote AP.* Put Subtangent $PT = a$, $AD = b$, $AP = x$, $PM = y$. By the Nature of the Curve $\dot{x} = \frac{a\dot{y}}{y}$: Whence $\dot{s} = cy\sqrt{\frac{aa\dot{y}^2}{y2} + \dot{y}^2} = c\dot{y}\sqrt{aa+yy}$; and by Form 9 and 13) $s = \frac{1}{2}y\sqrt{aa+yy} + \frac{1}{2}caa \times 2.30258$ Log. $y + \sqrt{aa+yy}$; but in D, $s = 0$, and $y = b$; therefore the Fluent corrected by Prop. XII. is $s = \frac{1}{2}y\sqrt{aa+yy} - \frac{1}{2}b\sqrt{aa+bb} + \frac{1}{2}caa \times 2.30258$ Log. $\frac{y+\sqrt{aa+yy}}{b+\sqrt{aa+bb}}$.

Ex. 13.

129. *Let the Cycloid AM revolve about RG*; Let $AP = x$, $PM = y$, Curve $AM = z$. By the Proceſs Ex. 8. Prob. VIII. $\dot{z} = \dot{x}\sqrt{\frac{a}{x}}$, thererefore $\dot{s} = c \times PR \times \dot{x}\sqrt{\frac{a}{x}} = \overline{a-x} \times c\dot{x}\sqrt{\frac{a}{x}} = ca^{\frac{3}{2}}x^{-\frac{1}{2}}\dot{x} - ca^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}$, whence $s = 2ca^{\frac{3}{2}}x^{\frac{1}{2}} - \frac{2}{3}ca^{\frac{1}{2}}x^{\frac{3}{2}} = c\sqrt{ax} \times \overline{2a - \frac{2}{3}x}$. And when $x = a$, the whole Surface deſcribed by $AMC = \frac{4}{3}caa$.

Ex. 14.

119. *Suppoſe the Catenary AM to revolve round the Axis AP*, by the Nature of the Curve $zz = 2ax + xx$, or $a + x = \sqrt{aa+zz}$, whence $\dot{x} = \frac{z\dot{z}}{\sqrt{aa+zz}}$; likewiſe $\dot{y} = \sqrt{\dot{z}^2 - \dot{x}^2} = \frac{a\dot{z}}{\sqrt{aa+zz}}$. Since $\dot{s} = cy\dot{z}$, aſſume the Fluent $s = cyz - t$ (by Rule 8. Prop. X.) This put into Fluxions is $\dot{s} = cy\dot{z} + cz\dot{y} - \dot{t}$, or $\dot{t} = cz\dot{y} =$

$ca z\dot{z}$

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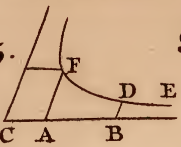

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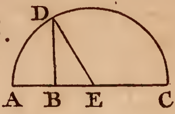

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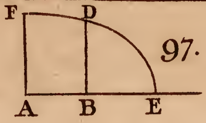

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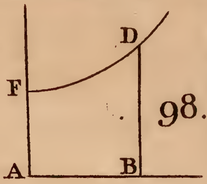

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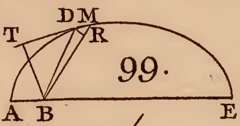

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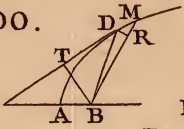

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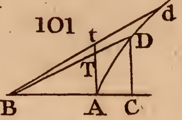

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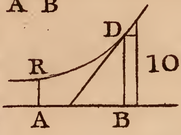

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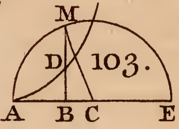

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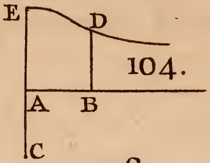

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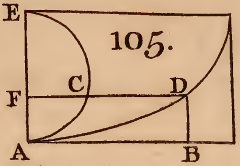

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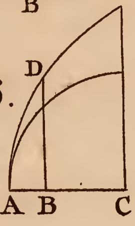

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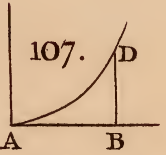

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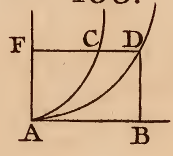

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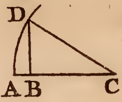

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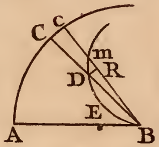

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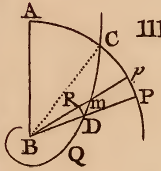

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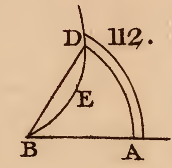

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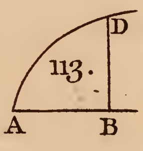

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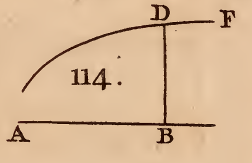

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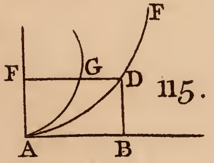

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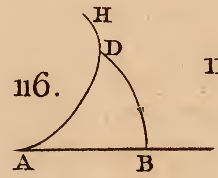

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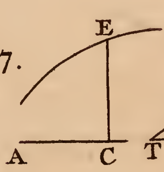

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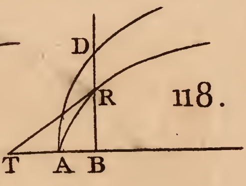

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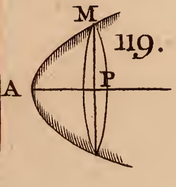

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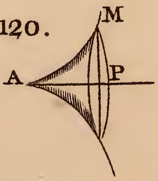

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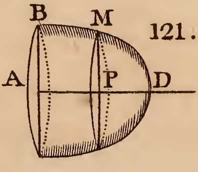

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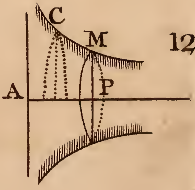


$\frac{caz\dot{z}}{\sqrt{aa+zz}}$, and (by Form the 3d) $t = ca\sqrt{aa+zz}$; whence $s = cyz - ca\sqrt{aa+zz} = cyz - caa - cax$. And when corrected $s = cyz - cax$.

PROB. XIV.

To find the Content of ſolid Bodies.

In any Solid *AMm*, generated by the Space *APM* 130.
revolving round the Axis *AP*; ſuppoſe the Plane *Mm* to move along the Axis *AP*, and by that Motion to deſcribe or generate that Solid; and ſuppoſe a given Rectangle *B* to move with the ſame Motion along the ſame Axis *AP*, and by that Motion to generate a Parallelopipedon; the Fluxions of theſe Solids will be as the deſcribing Planes *Mn* and *B* drawn into the Velocities of their Motions or the Fluxions of the Abſciſſa *AP*. Now ſince $B \times AP =$ Parallelopipedon, and $B \times$ Fluxion of *AP* = its Fluxion, conſequently the Fluxion of the Solid *AMm* = deſcribing Plane *Mm* × into the Fluxion of *AP*.

Likewiſe if the Solid *AMh* be ſuppoſed to be generated by a cylindric Surface *MHhm* continually ex- 131.
panding itſelf and moving along the Ordinate *PM*, ſtill retaining *AP* for its Axis; it may be the ſame way proved, that the Fluxion of the Solid *AMHhmA* or *HMEmH* generated thereby is = cylindric Surface *MHhm* multiplied by the Fluxion of the Ordinate *PM*. Therefore,

To find the Solidity of a Body, let the Abſciſſa $AP = x$ *Ordinate* $PM = y$, $MH = u$, $c = 3.1416$ *&c.* $s =$ *ſolid Content. Then by the Equation of the Curve expunge ne of the indetermined Quantities (and its Fluxion) ut of the Equation* $\dot{s} = cyy\dot{x}$ *for the Solid AMm, or out*

FIG. *of the Equation* $\dot{s} = 2cuy\dot{y}$ *for the cylindrical Solid* *AMHbm or HMEmb*; *and then find the Fluent.*

EXAMPLE 1.

123. *Let ABD be a Cone*, Height $AC = a$, CB or CD $= b$, then $bx = ay$; therefore $\dot{s} = cyy\dot{x} = \frac{cbbx^2\dot{x}}{aa}$, and $s = \frac{cb^2x^3}{3aa} = \frac{cyyx}{3}$, and the whole Cone $=$ $\frac{cabb}{3} =$ Baſe $\times \frac{1}{3}$ Height.

Ex. 2.

132. *Let BF be a Priſmoid, whoſe Baſes are right angle* *Parallelograms, though not ſimilar.*

Let its perpendicular Height $AL = b$, $AP = x$, $AD = d$, $BE = s$, $AG = n$, $BC = m$, by ſimilar Triangle $b : x :: s-d : \frac{s-d}{b} x = QI$, and $b : x :: m-n : \frac{m-n}{b}$ $= IK$; then $KI = d + \frac{s-d}{b} x$, and $IH = n +$ $\frac{m-n}{b} x$; then $\dot{s} = \overline{d + \frac{s-d}{b} x} \times \overline{n + \frac{m-n}{b} x} \times \dot{x}$ $dn\dot{x} + \frac{m-n}{b} dx\dot{x} + \frac{s-d}{b} nx\dot{x} + \overline{s-d} \times \frac{m-n}{bb} x^2\dot{x}$ therefore $s = dnx + \frac{m-n}{2b} dx^2 + \frac{s-d}{2b} nx^2 +$ $\frac{\overline{s-d} \times \overline{m-n}}{3bb} x^3$ for the Solid *HIKF*. And when $x =$ then the whole Solid $BF = dm + sn + 2sm + 2dn$ $\times \frac{b}{6} = : \overline{s + \frac{1}{2}d} \times m + \overline{d + \frac{1}{2}s} \times n : \times \frac{1}{3}b$.

Ex. 3.

130. *Let AMm be the Segment of a Sphere*, its Diamete $= a$, then $yy = ax - xx$, therefore $\dot{s} = cyy\dot{x} = cax$

$-cx^2\dot{x}$. Therefore $s = \frac{cax^2}{2} - \frac{cx^3}{3}$, for the Segment *AMPm*. And when $x = a$, the whole Sphere $= \frac{ca^3}{6} = \frac{2}{3}$ the circumſcribing Cylinder.

Ex. 4.

Let AMm be the Segment of a Sphere of an exceeding ſmall Height, here $yy = ax$ nearly, therefore (by Cor. 3. Prop. II.) $\dot{s} = cyy\dot{x} = cax\dot{x}$; therefore $s = \frac{caxx}{2} = \frac{cxy^2}{2}$, nearly.

Hence in any Solid of a finite Curvature, the Content of a Segment of a very ſmall Height, will be found to be half the Baſe drawn into the Height.

Ex. 5.

Let the Solid be a Paraboloid, where $ax = yy$; then 130. $\dot{s} = cyy\dot{x} = cax\dot{x}$; therefore $s = \frac{caxx}{2} = \frac{cyyx}{2} = \frac{1}{2}$ Baſe × Height.

And in general if $x = y^m$, then $\dot{x} = my^{m-1}\dot{y}$; and then $\dot{s} = cyy\dot{x} = cmy^{m+1}\dot{y}$; and therefore $s = \frac{cm}{m+2}y^{m+2} = \frac{cm}{m+2}y^2x$. Therefore in the conic Parabola, where $m = \frac{1}{2}$, that is when the Curve is convex towards *AP*, then the Solidity $= \frac{cyyx}{5} = \frac{1}{5}$ Baſe × Height.

Ex. 6.

Let AmD be a Parabolic Spindle, generated by the 130. *Parabola AMD (whoſe Axis is DT) revolving round the Ordinate AT.*

Let $AT = b$, $DT = d$, $DH = v$; $MH = u$, then *MP* or $y = d - v$, and $av = uu$ by the Nature of the Curve:

Whence $\dot{y} = -\dot{v} = \frac{-2u\dot{u}}{a}$; wherefore $\dot{s} = 2cuy\dot{y} = \frac{-4cu^2y\dot{u}}{a} = \frac{-4cu^2\dot{u}}{a} \times \overline{d - \frac{uu}{a}}$. Therefore the Fluent $s = \frac{-4cdu^3}{3a} + \frac{4cu^5}{5aa}$: but in A where s is 0, $u = b$; therefore the Fluent corrected is $s = \frac{4cdb^3}{3a} - \frac{4cdu^3}{3a} + \frac{4cu^5}{5aa} - \frac{4cb^5}{5aa} = \frac{8cdb^3}{15a} - \frac{4cu^3}{15a} \times \overline{5d - 3v}$, for the Solid *AMHbm*; and when v and $u = 0$, the whole Solid $ADd = \frac{8cdb^3}{15a} = \frac{8cbdd}{15} = \frac{8}{15}$ the Baſe × Height.

Ex. 7.

130. *Let AMm be an Hyperboloid*, Tranſverſe $= a$, Conjugate $= b$. Then $yy = \frac{bb}{aa} \times \overline{ax + xx}$; and $\dot{s} = cyy\dot{x} = \frac{cbb\dot{x}}{aa} \times \overline{ax + xx}$. And the Fluent $s = \frac{cbbxx}{2a} + \frac{cbbx^3}{3aa} =$ (expunging bb) $\frac{3ax + 2xx}{6a + 6x} \times cyy = \frac{3a + 2x}{6a + 6x}$ × Baſe × Height.

Hence if $x = 0$, then the Solid $= \frac{1}{2}$ Baſe × Height, and if x be infinite, then the Solid $= \frac{1}{3}$ Baſe × Height: Therefore the Hyperboloid is always between $\frac{1}{2}$ and $\frac{1}{3}$ the circumſcribing Cylinder, and is nearly $= \frac{5}{12}$ thereof.

Ex. 8.

133. *Let the Hyperbola CM revolve round the Conjugate AP*, A being the Center, $AC = b$, Semiconjugate $= a$. By the Nature of the Figure $yy = bb + \frac{bb}{aa}xx$;

whence $\dot{s} = cyy\dot{x} = cbb\dot{x} + \frac{cbbx^2\dot{x}}{aa}$, and $s = cbbx + \frac{cbb}{3aa}x^3 = \frac{2}{3}cbbx + \frac{1}{3}cyyx$, the Solid deſcribed by *ACMP*.

Ex. 9.

134. *Let the Hyperbola DM revolve round the Aſſymptote AP*, let $AB = b$, $BD = d$, $xy = aa$. Then $\dot{s} = cyy\dot{x} = \frac{ca^4\dot{x}}{xx}$, whence $s = \frac{ca^4}{x}$, let the Solid begin at B; and then the Fluent corrected is $s = \frac{ca^4}{b} - \frac{ca^4}{x} = caa \times \overline{d-y}$.

Note if *CAB* is not a right Angle, s muſt be diminiſhed in the Ratio of the Radius to the Sine of that Angle. When $y = 0$, the infinitely long Solid $DMPB = caad = cddb =$ Baſe × *AB*.

Or thus: Let $AP = x$, $PM = y$, $aa = xy$, then $u = x = \frac{aa}{y}$, whence $\dot{s} = 2cuy\dot{y} = 2caa\dot{y}$; therefore $s = 2caay = 2cyyx$, for the infinitely long Solid *AHMQP*.

Ex. 10.

128. *Let DM the Log. Curve revolve round the Aſſymptote AP*, let Subtangent $TP = a$, then $y\dot{x} = a\dot{y}$; and $\dot{s} = cyy\dot{x} = cay\dot{y}$; therefore $s = \frac{cayy}{2} =$ Baſe × $\frac{1}{2}$ *TP*, for the infinitely long Solid *MPAD*.

Ex. 11.

135. *Let AM be a Spheroid, A* the Center, $AE = a$, $AC = b$, then $yy = bb - \frac{bb}{aa}xx$; whence $\dot{s} = cyy\dot{x} = cbb\dot{x} - \frac{cbbx^2\dot{x}}{aa}$; therefore $s = cbbx - \frac{cbb}{3aa}x^3 = \frac{2}{3}cbbx + \frac{1}{3}cyyx$. And when $x = a$, then $s = \frac{2}{3}cbba = \frac{2}{3}$ Baſe × Height.

Ex.

EX. 12.

FIG. 136. *Let the Segment ADMN of a Circle revolve about the Sine AN*; $CD=r$, $CA=b$, AP or $CQ=x$, $PM=y=\sqrt{\overline{rr-xx}}-b$; then $\dot{s}=cyy\dot{x}=c\dot{x}\times\overline{rr+bb-xx}-2bc\dot{x}\sqrt{\overline{rr-xx}}$; and the Fluent $s=cx\times\overline{rr+bb-\frac{1}{3}xx}-2bc\times$ Area $CDMQ$ the Solid deſcribed by $ADMP$.

EX. 13.

137. *In the Ciſſoid AM revolving round* $AR=a$, we have $y=\frac{xx}{\sqrt{ax-xx}}$, whence $\dot{s}=cyy\dot{x}=\frac{cx^3\dot{x}}{a-x}=-cx^2\dot{x}-cax\dot{x}-caa\dot{x}+\frac{ca^3\dot{x}}{a-x}$: Therefore (by Form 4 &c) $s=-\frac{cx^3}{3}-\frac{cax^2}{2}-caax-2.30258ca^3\times\text{Log.}\,\overline{a-x}$: And when duly corrected $s=-\frac{1}{3}cx^3-\frac{1}{2}cax^2-caax+2.30258ca^3\,\text{Log.}\,a:-2.30258ca^3\,\text{Log.}\,\overline{a-x}=2.302585ca^3\times\text{Log.}\,\frac{a}{a-x}-\frac{1}{3}cx^3-\frac{1}{2}cax^2-caax$.

EX. 14.

138. *Let the Ciſſoid revolve round the Aſſymptote AP*, then $x=\frac{\overline{a-y}|^{\frac{3}{2}}}{\sqrt{y}}=u$: Whence $\dot{s}=2cuy\dot{y}=2cxy\dot{y}=2cy^{\frac{1}{2}}\dot{y}\times\overline{a-y}|^{\frac{3}{2}}$: Therefore (by Forms 10, 11 and 13) $s=ca\times$ Segment $ACD+\frac{2}{3}c\times\overline{ay-yy}|^{\frac{3}{2}}$, for the infinitely long Solid $ADMQ$. And the whole $ABMQ$ $=ca\times$ Semicircle ACB.

EX. 15.

139. *Let the Conchoid DM revolve about the Aſſymptote AP*; $AD=a$, $AC=b$, then $\overline{b+y}\sqrt{\overline{aa-yy}}=xy$, and $u=x$

therefore $\dot{s} = 2cuy\dot{y} = 2cxy\dot{y} = 2c\dot{y} \times \overline{b+y}\sqrt{aa-yy} = 2bc\dot{y}\sqrt{aa-yy} + 2cy\dot{y}\sqrt{aa-yy}$. Describe the Quadrant *ADK*; then (by Forms 10 and 13 &c) $s = 2cb \times$ Area $AKLR - \frac{2}{3}c \times \overline{aa-yy}|^{\frac{3}{2}}$. And, duly corrected, $s = 2cb \times$ Area $AKLR + \frac{2}{3}ca^3 - \frac{2}{3}c \times \overline{aa-yy}|^{\frac{3}{2}}$, for the infinitely long Solid *ARMQP*. FIG.

Ex. 16.

Let the Conchoid DM revolve round the Axis DR, here the Circle described by $RM = cxx$, and the Fluxion of *DR* or $a-y$ is $-\dot{y}$; therefore the Fluxion of the Solid $\dot{s} = -cxx\dot{y} = -c\dot{y} \times \frac{\overline{b+y}|^2}{yy} \times \overline{aa-yy} = c\dot{y} \times \overline{b+y}|^2 - \frac{aabbc\dot{y}}{yy} - \frac{2aabc}{y}\dot{y} - aac\dot{y}$: Whence $s = \frac{c}{3} \times \overline{b+y}|^3 + \frac{cbbaa}{y} - 2aabc \times 2.302585 \times$ Log. y :$-aacy$. And by due Correction (when $y=a$) $s = \frac{1}{3}c \times \overline{b+y}|^3 - \frac{1}{3}c \times \overline{b+a}|^3 + \frac{aabbc}{y} - abbc + a^3c - aacy + 2.30258 \times 2caab \times$ Log. $\frac{a}{y}$. 139.

Ex. 17.

Let the Cycloid AMG revolve about the Axis RG, let $AR=a$, $AP=x$, $RP=u$, $pM=y$, Arch $AB=v$, $PB=z$, then $x=a-y$, and $\dot{x}=-\dot{y}$, and by the Nature of the Figure $u=v+z$, and $\dot{u}=\dot{v}+\dot{z}=\frac{a\dot{x}}{2\sqrt{ay-yy}}+\frac{2y-a}{2\sqrt{ay-yy}}\dot{x}=\dot{x}\sqrt{\frac{y}{a-y}}=-\dot{y}\sqrt{\frac{y}{a-y}}$: then $\dot{s}=2cuy\dot{y}$; and assume $s=cuyy+t$, this put into Fluxions we shall find $\dot{t}=-cyy\dot{u}=\frac{cy^{\frac{5}{2}}\dot{y}}{\sqrt{a-y}}$; whence 129.

whence by Forms the 10th and 11th) $t = \frac{5ca^3}{16}\varphi - \frac{8yy + 10ay + 15aa}{24} \times c\sqrt{ay - yy}$ (putting $\varphi = \frac{8 \text{ Sectors } CBR}{aa}$): Therefore $s = cuyy + \frac{5}{2}ca \times$ Sector $CBR - \frac{15caa + 10acy + 8cyy}{24}\sqrt{ay - yy}$, for the Solid deſcribed by $PMGR$. And when $y = a$, then $s = \frac{5}{2}ca \times$ Semicircle ABR, the Solid deſcribed by the whole Space $AMGR$ revolving about RG.

Ex. 18.

130. *Let the Catenary AM revolve about the Axis AP*; here $AP = x$, $PM = y$, $AM = z$, and $zz = 2ax + xx$, whence (ſee Ex. 8. Prob. III.) $z\dot{y} = a\dot{x}$. Now $\dot{s} = cyy\dot{x}$, and aſſume $s = cyyx + t$, this in Fluxions gives $\dot{t} = -2cxy\dot{y} =$ (becauſe $\dot{y} = \frac{a\dot{x}}{z}$) $-2cay \times \frac{x\dot{x}}{z} =$ (becauſe $x\dot{x} = z\dot{z} - a\dot{x}$) $-2cay \times \overline{\dot{z} - \frac{a\dot{x}}{z}} = -2cay \times \overline{\dot{z} - \dot{y}} = 2cay\dot{y} - 2cay\dot{z}$; then aſſume $t = cayy - 2cayz + u$, this Equation in Fluxions will produce $\dot{u} = 2caz\dot{y} = 2caa\dot{x}$, whence $u = 2caax$. Therefore the Solid $s = cxyy + cayy - 2cayz + 2caax$.

Ex. 19:

139. *Let the Abſciſſa* $AP = x$, *Ordinate* $PM = y$, $p = 3.1416$; *to find the Solidity of a Body deſcribed by the Revolution of any Curve defined by this Equation* $yy = A + Bx + Cx^2$.

The Fluxion of the Solid $pyy\dot{x} = \overline{A + Bx + Cx^2} \times p\dot{x}$, and the Fluent is $\overline{A + \frac{Bx}{2} + \frac{Cx^2}{3}} \times px$ for the Solidity; out of this expunge either $\frac{Bx}{2}$ or $\frac{Cx^2}{3}$ by the Equation of the Curve, and you will

have

FIG. 130.

have the Solid $= \overline{yy + A - \tfrac{1}{3}Cxx} : \times \frac{px}{2} = \overline{yy + 2A + \tfrac{1}{2}Bx} : \times \frac{px}{3}$, which contains two general Theorems for all Solids of this Kind. Therefore from the Equation of the Curve get the Values of A, B, C, and ſubſtitute them in that Expreſſion of the Solid where you have the feweſt of the Quantities A, B, C, as in the following Caſes.

1. *Let* $y = ax$ *denote a Cone,* then $yy = aaxx$, and you have $A = 0$, $B = 0$, $C = aa$. Whence the Solid $= yy \times \frac{px}{3}$.

2. *In the Fruſtrum of a Cone,* $b =$ Radius of the Top, and $y - b = ax$, or $yy = bb + 2abx + aaxx$; here $A = bb$, $B = 2ab$, $C = aa$, and the Solid $= \overline{yy + 2bb + bax} \times \frac{px}{3} =$ (expunging ax) $\overline{yy + by + bb} \times \frac{px}{3}$.

3. *For the Segment of a Sphere* whoſe Radius is $r.$ $yy = 2rx - xx$. Here $A = 0$, $B = 2r$, $C = -1$, and the Solid $= \overline{yy + \tfrac{1}{3}xx} \times \frac{px}{2} = \overline{yy + rx} \times \frac{px}{3}$.

4. *For the Fruſtrum of the Sphere,* $yy = rr - xx$; here $A = rr$, $B = 0$, $C = -1$, and then the Fruſtum $= \overline{yy + 2rr} \times \frac{px}{3}$.

5. *In the Spheroid,* $a =$ Tranſverſe, $d =$ Conjugate, and $yy = \frac{ddx}{a} - \frac{ddxx}{aa}$, here $A = 0$, $B = \frac{dd}{a}$, $C = \frac{-dd}{aa}$. Solid $= \overline{yy + \frac{ddx}{2a}} \times \frac{px}{3}$, for the Segment.

6. *For the middle Fruſtrum of the Spheroid,* $2t =$ Tranſverſe, $2c =$ Conjugate, then $yy = cc - \frac{cc}{tt}xx$,

FIG. 130. where $A = cc$, $B = 0$, $C = \frac{-cc}{tt}$, the Solid $= \overline{yy + 2cc} \times \frac{px}{3}$.

7. *In a Paraboloid* $yy = ax$, here $A = 0$, $B = a$, $C = 0$, the Solid $= yy \times \frac{px}{2}$.

8. *For the Fruſtrum of a Paraboloid*, $b =$ Radius of the leſſer Baſe, $yy = bb + ax$; here $A = bb$, $B = a$, $C = 0$, then the Solid $= \overline{yy + bb} \times \frac{px}{2}$.

9. *In the Hyperboloid*, $2a =$ Tranſverſe, $2d =$ Conjugate, and $yy = \frac{2ddx}{a} + \frac{dd}{aa}xx$; here $A = 0$, $B = \frac{2dd}{a}$, $C = \frac{dd}{aa}$. Solid $= \overline{yy + \frac{ddx}{a}} \times \frac{px}{3}$, the Segment.

PROB. XV.

140. *The Nature of the reflecting Curve AMn, and the luminous Point L being given; To find the Focus F, or the Concourſe of the neareſt reflected Rays MF, nF.*

Take the Particle of the Curve *Mn* infinitely ſmall, and let *C* be the Center and *CM* the Radius of Curvature of the Arch *Mn*: and on *ML*, *nL*, *MF*, *nF* let fall the Perpendiculars *CE*, *Ce*, *CG*, *Cg*; alſo on the Centers *L*, *F*, deſcribe the ſmall Arches *Mr*, *no*; then the little Triangles *Mon*, *Mnr* are equal and ſimilar, and $Mo = nr$. By the Nature of Reflection the Angle $LMC = CMF$, and $LnC = CnF$, whence $CE = CG$, and $Ce = Cg$. Now if $CE = Ce$ (that is, if *L* falls in *E* or *e*) then will $CG = Cg$; that is, the Point

FIG. 140. F will fall in G or g when M and n coincide: But if Ce be leſs than CE, that is, if L falls below E, then will Cg be leſs than CG, and the Interſection F will then fall above G towards M; and the contrary. The Triangles LEQ, LMr; and Fon, FSG are ſimilar, and $EQ = CE - Ce = CG - Cg = SG$; and $Mr = no$, and $FG = MG - MF = ME - MF$; therefore $LM : LE :: (Mr : EQ :: no : SG :: FM : FG ::)\ FM : ME - MF$. Whence $MF = \frac{LM \times ME}{2LM - ME}$.

Otherwiſe thus: Draw the Tangent MP, and the Perpendicular LP, and let $LM = y$, $LP = u$, $ME = v$, and by Prob. V. $MC = \frac{y\dot{y}}{\dot{u}}$, whence by ſimilar Triangles $vy = \frac{yu\dot{y}}{\dot{u}}$, or $v\dot{u} = u\dot{y}$, therefore $MF = \frac{uy\dot{y}}{2y\dot{u} - u\dot{y}}$.

1. *Wherefore if CM be the Radius of Curvature, CE perpendicular to LM, and LP perpendicular to the Tangent PM, and we make the Diſtance of the radiating Point $LM = y$, $ME = v$, $LP = u$: Then compute he Value of v by Prob. V. and take $MF = \frac{vy}{2y - v}$: And when AM is convex towards L write $-v$ inſtead f $+v$.*

2. *Or find u from the Nature of the Curve, by Help of which expunge $\dot{u}$ out of the Equation $MF = \frac{uy\dot{y}}{2y\dot{u} - u\dot{y}}$.*

Cor. The Curve FfH paſſing through all the Points F, or touching all the reflected Rays MF, mf s called the Catacauſtic or Cauſtic by Reflexion. In which any Portion HfF of the Curve is $= LM + MF - LA - AH$. For drawing mL infinitely near mL, nd Mo, mr perpendicular to mF, ML; then ſince $Mo = Mr$, therefore $Lm + mf = LM + Mf$, or $LM +$ 141.

FIG. $Mf - Lm - mf = 0$; and adding fF, $LM + MF - Lm - mf = fF$; but theſe Moments are as the Fluxions, and therefore the Fluents thereof will be equal, that is the Curve $HF = LM + MF - LA - AH$.

EXAMPLE I.

142. *Let AM be a right Line*, then v is infinite, whence $MF = \frac{vy}{2y - v} = \frac{vy}{-v} = -y$.

Or thus: u is a ſtanding Quantity and $\dot{u} = 0$, therefore $MF = \frac{uy\dot{y}}{2y\dot{u} - u\dot{y}} = \frac{uy\dot{y}}{-u\dot{y}} = -y$: Whence Perpendicular $PF = PL$, and F is the Focus of the reflected Rays.

Ex. 2.

143. *Let MD be a Circle, C* the Center; then $MF = \frac{vy}{2y - v}$. And when y is infinite, $MF = \frac{1}{2}v = \frac{1}{2}ME$. And if MD be very ſmall, $MF = \frac{LD \times DC}{2LC + CD}$. And if $LC = CD$, then $MF = \frac{2}{3}v = \frac{2}{3}ME = \frac{1}{3}LM$.

Ex. 3.

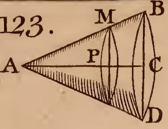
144. *Let MD be a Parabola*, and let the Rays be parallel to the Axis DB, then y is infinite, whence $MF = \frac{vy}{2y - v} = \frac{1}{2}u$. But by Prob. V. $v = \frac{2 . MQ^2}{r}$ (putting r = Latus Rectum) alſo $QT = \frac{2QM^2}{r}$, therefore $QT = v$, and $QF = \frac{1}{2}v = MF$, and therefore F is the Focus of the Parabola.

Ex. 4.

145. *Let DM be a Parabola*, and let all the Rays be perpendicular to the Axis DQ. Let $DP = x$, $PM = z$, $MQ = s$, and $rx = zz$: Then by Prob. V. ME or v

$=$

123.



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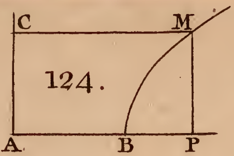

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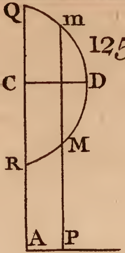

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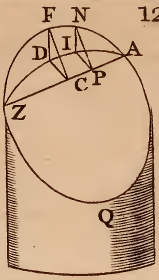

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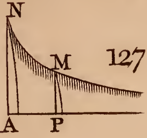

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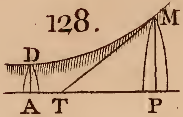
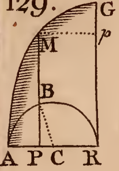


129.



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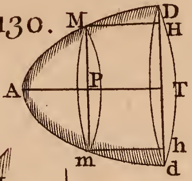

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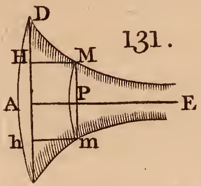

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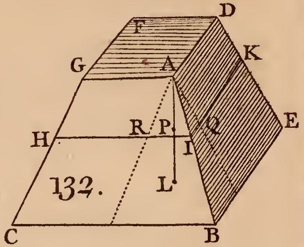

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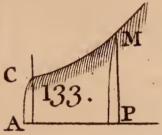

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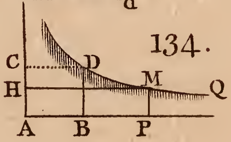

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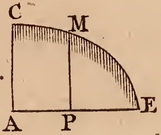

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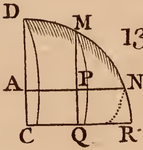

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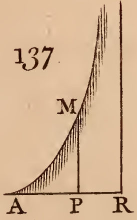

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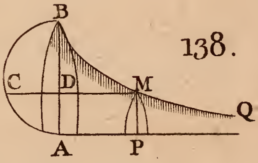

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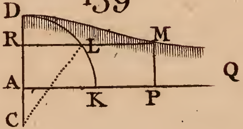

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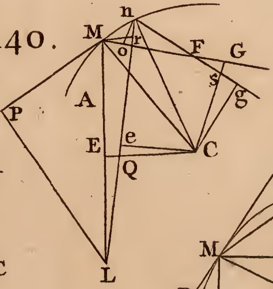

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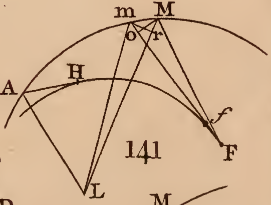

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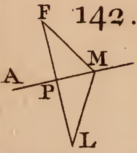

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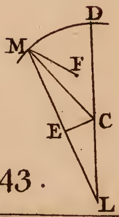

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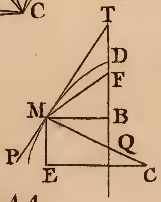

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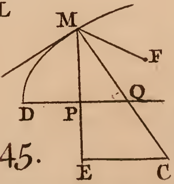

Pl. VI. *p.* 292

$= \frac{4ssz}{rr}$, and $ss = zz + \frac{1}{4}rr$ by the Nature of the Figure: Whence $MF = \frac{vy}{2y - v} = \frac{1}{2}v = \frac{2z^3}{rr} + \frac{1}{2}z = \frac{2zx}{r} + \frac{1}{2}z$.

Ex. 5.

Let DM be the Log. Spiral, the Center L the luminous Point. By Prob. V. $v = y$, whence $MF = \frac{vy}{2y - v} = y$. 146.

Ex. 6.

Suppoſe DM to be an Ellipſis, L the Focus, the Tranſverſe $= 2r$, Conjugate $= 2c$; then by the Property of the Curve PL or $u = \frac{cy}{\sqrt{2ry - yy}}$, and $\dot{u} = \frac{cry\dot{y}}{\overline{2ry - yy}|^{\frac{3}{2}}}$; whence $MF = \frac{uy\dot{y}}{2y\dot{u} - u\dot{y}} = \frac{cy^2\dot{y} \times \overline{2ry - yy}}{cy^3\dot{y}} = 2r - y$; conſequently (ſince the Angle $LMP = FMT$) the Point F is the other Focus. 147.

And in like Manner it will be found that Rays iſſuing from one Focus of an Hyperbola, and reflected by the Curve will diverge from the other Focus.

PROB. XVI.

The Nature of the refracting Curve AMn, and the luminous Point L being given; to find the Focus F, or the Point where the neareſt refracted Rays MF, nF concur; and where they meet the Axis of the Figure. 148.

Suppoſe the Arch Mn to be infinitely ſmall, and let C be the Center and CM the Radius of Curvature in

FIG. in M; and let fall the Perpendiculars CE, Ce on the Rays of Incidence LM, Ln, and the Perpendiculars CG, Cg on the refracted Rays MF, nF; and let the Sine of Incidence CE to the Sine of Refraction CG be as m to n. On the Centers L, F, deſcribe the ſmall Arches Mr, Mo. Now ſince Ce exceeds CE, therefore Cg exceeds CG, they being in a given Ratio; whence MF, nF interſect beyond G.

The Figures $GCME$ and $onMr$ are ſimilar; therefore $ME : MG :: Mr : Mo = \frac{MG \times Mr}{ME}$; and $LM : LQ :: Mr : Qe = \frac{LQ \times Mr}{LM}$: And by the Property of Refraction, $m : n :: (Ce : Cg :: CE : CG :: Ce - CE : Cg - CG ::)\ eQ : Sg = \frac{n \times LQ \times Mr}{m \times LM}$; and by the ſimilar Triangles FoM, Fgs; $Mo - Sg : Mo :: MG : MF = \frac{m \times MG^2 \times LM}{m \times MG \times LM - n \times LQ \times ME}$.

149. Produce MF till it interſect the Axis of the Curve in O, and let $LA = d$, $AO = f$, $LM = y$, $MO = s$, $AH = x$, $MH = z$, then $y = \sqrt{z^2 + \overline{d+x}^2}$, and $s = \sqrt{zz + \overline{f-x}^2}$. And it is $\acute{y} : -\acute{s} :: rn : on :: m : n$,

148. and $n\acute{y} = -m\acute{s}$ or $n\dot{y} = -m\dot{s}$, that is $\frac{nz\dot{z} + nd\dot{x} + nx\dot{x}}{\sqrt{zz + \overline{d+x}^2}}$

$$= \frac{mf\dot{x} - mx\dot{x} - mz\dot{z}}{\sqrt{zz + \overline{f-x}^2}}.$$ Therefore,

149. 1. *To find the Focus F, let CM be the Radius of Curvature, CE perpendicular to LM, and CG to MG, $ME = v$, $MG = u$. $LM = y$, m and n the Sines of Incidence and Refraction. Then find v and u by Prob. V. and take* $MF = \frac{muuy}{muy - nvv - nvy}$. *If AM is con-*

cave towards L, write $-v$ *for* v, *and* $-u$ *for* u: *And if the Rays converge when they fall on the Curve AM, write* $-y$ *for* y.

2. *To find the Point O where the refracted Ray meets the Axis of the Curve, let* $LA = d$, $AO = f$, $AH = x$, $MH = z$, *then by the Nature of the Curve expunge* $\dot x$ *or* $\dot z$ *out of the Equation*

$$\frac{nz\dot z + nd\dot x + nx\dot x}{\sqrt{zz+\overline{d+x}^2}} = \frac{mf\dot x - mx\dot x - mz\dot z}{\sqrt{zz+\overline{f-x}^2}};$$

and by Reduction find f: And when the C is concave towards L, write $-x$, *for* $+x$, *&c.*

150. Cor. The Curve NfF paſſing through all the Points F, or which touches all the refracted Rays KN, MF, is called the Diacauſtic or Cauſtic by Refraction. And any Portion of it $NF = FM + \frac{n}{m}ML - NK - \frac{n}{m}KL$. For ſuppoſing Mn infinitely ſmall, and drawng Mo, Mr, Perpendiculars on fn, nL; then by he Nature of Refraction $m : n :: rn : on = \frac{n}{m}rn$, herefore $on = \frac{n}{m}rn = 0$; that is $MF - nf - \frac{n}{m}\times\overline{Ln - LM} = 0$, and adding Ff; we have $Ff = MF - nf - \frac{n}{m}\times\overline{Ln - LM}$; but theſe Moments are s the Fluxions; whence the Fluents will be equal, or $FN = FM - KN - \frac{n}{m}\times\overline{LK - LM}$.

EXAMPLE 1.

FIG. 151. *Let AM be a plane Surface,* then v, u are infinite whence $\frac{muuy}{muy - nvv - nvy} = \frac{muuy}{-nvv}$. Now le $r =$ the infinite Radius of Curvature, the Perpendicular $LA = p$, then by the ſimilar Triangles LAM MCE, $v = \frac{rp}{y}$; alſo $mm \times \overline{rr - uu} = nn \times \overline{rr - vv}$ by the right angled Triangles MEC, MGC: Whence $uu = \frac{mm - nn}{mm} rr + \frac{nn}{m^2} vv$; therefore $MF =$ $\frac{-muuy}{nvv} = -\frac{mm - nn}{mnvv} \times rry - \frac{ny}{m} = -\frac{m^2 - n^2}{mnp^2} y^3$ $-\frac{n}{m} y$. And near the Point A, $Af = -\frac{m}{n} p$.

Ex. 2.

152. *Let parallel Rays fall on the convex Side of the Spere* AM; then y is infinite, and $MF = \frac{muuy}{muy - nvv - nvy}$ $= \frac{muu}{mu - nv}$, and near the Vertex A, $u = v = AC$ whence $Af = \frac{m \times AC}{m - n}$.

Ex. 3.

153. *Let parallel Rays fall on the concave Side of the Sphere* AM; then y is infinite, and v, u are negative whence $MF = \frac{muuy}{-muy - nvv + nvy} = \frac{muu}{nv - mu}$, and at the Vertex A, $v = u = AC$, then $AF = \frac{mu}{n - m} =$ $\frac{m}{n - m} AC$.

Ex. 4.

FIG. 154. *Suppoſe the Rays proceeding from L, to fall on the convex Side of the Sphere AM*; then $MF = \frac{muuy}{muy-nvv-nvy}$. And in or near the Vertex, $MF = \frac{muy}{\overline{m-n}\times y-nu} = \frac{mry}{\overline{m-n}\times y-nr}$.

And to find where the refracted Ray meets the Axis of the Sphere, $zz=2rx-xx$, and $z\dot{z}=r\dot{x}-x\dot{x}$, by which expunge z and $\dot{z}$ out of the Equation $\frac{nz\dot{z}+nd\dot{x}+nx\dot{x}}{\sqrt{zz+\overline{d+x}^2}} = \frac{mf\dot{x}-mx\dot{x}-mz\dot{z}}{\sqrt{zz+\overline{f-x}^2}}$, and there ariſes $\frac{nr+nd}{\sqrt{2rx+2dx+dd}} = \frac{mf-mr}{\sqrt{2rx-2fx+ff}}$, by Reduction of which f is found. And in the Vertex A where x is 0, $\frac{mf-mr}{f} = \frac{nr+nd}{d}$, or $f = \frac{mdr}{\overline{m-n}\times d-nr}$.

Ex. 5.

155. *If Rays fall on the concave Surface of the Sphere AM*, then $MF = \frac{muuy}{-muy-nvv+nvy}$. And near the Vertex, $Mf = \frac{muy}{\overline{n-m}\times y-nu} = \frac{mry}{\overline{n-m}\times y-nr}$.

To find where MF cuts the Axis, here $zz=2rx-xx$; put the Equation into Fluxions, and write $-x$ for x, $-\dot{x}$ for $\dot{x}$, $-r$ for r in the Equation $\frac{nz\dot{z}+nd\dot{x}+nx\dot{x}}{\sqrt{zz+\overline{d+x}^2}} = \frac{mf\dot{x}-mx\dot{x}-mz\dot{z}}{\sqrt{zz+\overline{f-x}^2}}$, and we have $\frac{nr-nd}{\sqrt{dd-2dx+2rx}} = \frac{-mr-mf}{\sqrt{ff+2fx+2rx}}$; and in the Vertex $\frac{nr-nd}{d} = \frac{-mr-mf}{f}$, or $f = \frac{mrd}{\overline{n-m} \times d-nr}$

155. COR. *Hence we may find if the Sphere in this Caſe has a geometrical Focus*, or ſuch a one where all Rays are refracted accurately to one Point. When this happens, *AO* muſt be a determined Quantity for all Parts of the Curve, the ſame as at the Vertex; and therefore the Powers of x muſt vaniſh out of the general Equation expreſſing *AO*; conſequently the Coefficients of each Power, being equated, muſt deſtroy one another. Thus the Equation for *AO* being $n^2 \times \overline{r-d}^2 \times \overline{ff+2fx+2rx} = m^2 \times \overline{r+f}^2 \times \overline{dd-2dx+2rx}$, then putting $n^2 \times \overline{r-d}^2 \times \overline{2f+2r} \times x = m^2 \times \overline{r+f}^2 \times \overline{2r-2d} \times x$, after reducing the Equation, there will be found $n^2 d = n^2 r - m^2 r - m^2 f = n^2 r - m^2 r - m^2 \times \frac{mrd}{nd-md-nr}$; from whence will be found $d = \frac{m+n}{n} r$, therefore $f = -\frac{m+n}{m} r$, which being negative, ſhows that the Focus *O* lies on the ſame ſide as the radiating Point *L*. Therefore when the Diſtance of the radiating Point is ſuch that $d = \frac{m+n}{n} r$, then *F* or *O* will be a geometrical Focus for all Rays falling on all Points of the Sphere; and d will be to f : : as m to n, and the Rays after Refraction will diverge from *O*.

And on the contrary Rays converging to *O*, and falling on the Sphere, will all be accurately refracted to their Focus *L*.

Ex. 6.

Let parallel Rays fall upon the Spheroid AM, let Tranſverſe $= b$, Parameter $= a$, then $zz = ax - \frac{a}{b}xx$, and $z\dot{z} = \frac{1}{2}a\dot{x} - \frac{a}{b}x\dot{x}$; and ſince d is infinite, therefore the Equation for AO becomes $n = \frac{mf - mx - \frac{ma}{2} + \frac{ma}{b}x}{\sqrt{ax - \frac{a}{b}xx + \overline{f - x}^2}}$, whence f will be found; and in the Vertex where $x = 0$, $mf - \frac{1}{2}ma = nf$, or $f = \frac{\frac{1}{2}ma}{m - n}$. FIG. 156.

Cor. *And to find if the Spheroid has a geometrical Focus,* we have from the above Equation, $n^2 \times : ff - 2fx + xx + ax - \frac{a}{b}xx = \overline{mf - \frac{ma}{2}}\Big|^2 \; \overline{2mf - ma} \times \overline{\frac{max}{b} - mx} + \overline{\frac{ma}{b} - m}\Big|^{2} x^2$, then equalling the Coefficients of x and x^2, $n^2 \times \overline{a - 2f} = \overline{2mf - ma} \times \overline{\frac{ma}{b} - m}$, alſo $nn \frac{-nna}{b} = \overline{\frac{ma}{b} - m}^2$. From the former Equation $2m^2 - 2n^2 - \frac{2m^2a}{b} \times f$, or $\overline{2m^2 - 2n^2 - \frac{2m^2a}{b}} \times \frac{\frac{1}{2}ma}{m - n} = m^2a - \frac{m^2a^2}{b} - n^2a$, hence we find $a = \frac{m^2 - n^2}{m^2}b$. And from the ſecond Equation, we get likewiſe $a = \frac{m^2 - n^2}{m^2}b$; therefore we may conclude that the Spheroid has a geometrical Focus, when $a = \frac{b \times \overline{m^2 - n^2}}{m^2}$; and then $f = \frac{\frac{1}{2}ma}{m - n} = \frac{m + n}{2m}b =$ 156.

156. $\frac{1}{2}b + \frac{bn}{2m}$; but $\frac{bn}{2m}$ = Diſtance of the Focus from the Center; therefore f = Diſtance of the remoter Focus from the Vertex. And therefore in this Caſe all parallel Rays falling on all Points of the Spheroid, will be accurately refracted to the further Focus of the Figure.

And on the contrary, Rays iſſuing from the further Focus of this Spheroid and refracted at the Surface, will all emerge parallel to the Axis.

PROB. XVII.

FIG. *To find the Center of Gravity of any Line, Surface or Solid.*

The Center of Gravity of a Body is that Point upon which, if it were ſuſpended, it would reſt in any given Poſition.

157. Let MN be any Figure or Solid Body, regular or irregular; C its Center of Gravity; and ſuppoſe it to be ſuſpended in C upon the horizontal Line SC, and the Point of Suſpenſion to be in S. Let all the infinitely ſmall Particles of the Body be reduced to the Line SC, ſituated reſpectively in Planes perpendicular to SC; as at e, d, b, &c. Now by Mechanics, the Force of any Particle e to turn the Body about C, is as its Weight and Diſtance from C, that is, as $Ce \times e$, or as $\overline{SC - Se} \times e$, for the ſame Reaſon the Force of all the particles between S and C will be $\overline{SC - Se} \times e + \overline{SC - SD} \times d + \overline{SC - Sb} \times b$, &c. In like Manner the Force of a Particle beyond C, as g, to turn the Body the contrary Way about C, is $Cg \times g$, or

$\overline{Sg-SC}\times g$; and the Force of all the Particles g, h, i, &c. will be $\overline{Sg-SC}\times g+\overline{Sh-SC}\times h+\overline{Si-SC}\times i$, &c. But as C is the Center of Gravity, the Forces on each Side will be equal; that is, $\overline{SC-Se}\times e+\overline{SC-Sd}\times d$, &c. $=\overline{Sg-SC}\times g+\overline{Sh-SC}\times h$, &c. and $SC\times e+SC\times d+SC\times g+SC\times h$, &c. $=Se\times e+Sd\times d+Sg\times g+Sh\times h$, &c. whence $SC=$ FIG. 157.

$$\frac{Se\times e+Sd\times d+Sb\times b+Sa\times a+Sg\times g+Sh\times h,\ \&c.}{e+d+b+a+g+h,\ \&c.\ =MN.}$$

Now if any one of the variable Diſtances as Sb be called x; the Body MN, S; then will $Sb\times b=x\dot{s}$; and the Sum of all the $Sa\times a+Sb\times b+Sd\times d$, &c. $=$ Sum of all the $x\dot{s}$, or the Fluent of $x\dot{s}$; and the Sum of all the $a+b+d$, &c. $=$ Sum of all the $\dot{s}$, or the Fluent of $\dot{s}$; that is the Body MN; therefore

$$SC=\frac{\text{Fluent of } x\dot{s}}{\text{Fluent of } \dot{s}}=\frac{\text{Fluent of } x\dot{s}}{\text{Body } MN}.$$ Therefore,

To find the Center of Gravity; let $s=$ Line, Surface or Solid: Multiply the Fluxion of the Line, Surface or Solid ($\dot{s}$) by the Diſtance (of the Center of Gravity of the generating Point, Line or Plane) from the Axis of Suſpenſion; and find the Fluent z; then $\frac{z}{s}=$ Diſtance of the Center of Gravity from the Point of Suſpenſion.

EXAMPLE I.

Let SB be a right Line or Cylinder, S the Point of Suſpenſion; $SB=x$; then $\dot{z}=x\dot{x}$, and $z=\frac{1}{2}xx$; therefore $\frac{z}{s}=\frac{1}{2}x$, for the Diſtance of the Center of Gravity SC. 158.

Ex. 2.

FIG. 159. *In the Triangle SQD*, whoſe Point of Suſpenſion is S; let SF biſſect the oppoſite Side QD, then the Center C is in the Line SF; draw AE parallel, and SG perpendicular to QD, put $SF=a$, $SB=x$, $SI=v$, $SG=b$, $QD=b$, $AE=y$. By ſimilar Triangles $v=\frac{bx}{a}$, and $\dot{v}=\frac{b\dot{x}}{a}$, $y=\frac{bx}{a}$; then $\dot{z}=xy\dot{v}=\frac{bbx^2\dot{x}}{aa}$; and $z=\frac{bbx^3}{3aa}$. Alſo $\dot{s}=y\dot{v}=\frac{bbx\dot{x}}{aa}$, and $s=\frac{bbx^2}{2aa}$: Therefore $\frac{z}{s}=\frac{2}{3}x$, and when $x=a$, $\frac{z}{s}=\frac{2}{3}a=SC$.

Ex. 3.

160. *Let AM be the Arch of a Circle*, s its Center, $AD=b$, $SE=r$, $SB=x$, $AM=v$, $BM=y$. It is evident the Center of Gravity of any Arch AEG is in the Line SE that biſſects it. Whence $\dot{z}=x\dot{v}=$ (by the Nature of the Circle) $-r\dot{y}$; and $z=-ry$; and the Fluent corrected is $z=br-ry$. Whence $\frac{z}{s}=\frac{rb-ry}{v}$, and when $y=0$, $\frac{z}{s}=\frac{rb}{v}=SC$, the Diſtance of the Center of Gravity of the Arch AEG from S.

Ex. 4.

161. *For the Sector of a Circle MmS*, whoſe Center and Point of Suſpenſion is S; let Arch $Mm=c$, Radius $SM=r$, $Mm=a$, $SQ=x$, $QDq=v$. Then $Qq=\frac{ax}{r}$, and by the laſt Example, the Diſtance of the Center of Gravity of the Arch QDq from S is $=\frac{ax^2}{rv}$, therefore $\dot{z}=\frac{ax^2\dot{x}}{r}$, and $z=\frac{ax^3}{3r}$; therefore

fore $\frac{z}{s} = \frac{ax^3}{3r \times \frac{1}{2}xv} = \frac{2ax^2}{3rv} = \frac{2ax}{3c}$; and when $x = r$, then $SC = \frac{2ar}{3c}$.

Ex. 5.

162. *For the Circular Area PQD*, Let SD or $SE = r$, $SQ = b$, $PQ = c$, $SB = x$, $PA = v$, $AB = y$, by the Nature of the Circle $y = \sqrt{rr - xx}$; then $\dot{z} = yx\dot{x} = x\dot{x}\sqrt{rr - xx}$, and $z = -\frac{1}{3} \times \overline{rr - xx}^{\frac{3}{2}}$; corrected $z = \frac{\overline{rr - bb}^{\frac{3}{2}} - \overline{rr - xx}^{\frac{3}{2}}}{3} = \frac{c^3 - y^3}{3}$. Alſo $s =$ Area $PABQ = \frac{vr + xy - cb}{2}$; whence $\frac{z}{s} = \frac{2}{3} \times \frac{c^3 - y^3}{vr + xy - cb}$ $= Sc$. And the Diſtance of the Center of Gravity of the whole $PADQ$ from S is $= \frac{2c^3}{3vr - 3cb}$.

Again in Reſpect of the Axis of Suſpenſion SD; ſince the Center of Gravity of the deſcribing Line y, is in the Middle of BA, therefore $\dot{z} = \frac{1}{2}y \times y\dot{x} = \frac{rr\dot{x} - x^2\dot{x}}{2}$, whence $z = \frac{r^2x}{2} - \frac{x^3}{3}$. But (in Q, $x = b$, $z = 0$) the Fluent corrected is $z = \frac{3rrx - 3rrb - x^3 + b^3}{6}$; therefore $\frac{z}{s} = \frac{1}{3} \times \frac{3r^2x - 3r^2b - x^3 + b^3}{vr + xy - cb}$; and when $x = r$, $\frac{2r^3 - 3r^2b + b^3}{3rv - 3cb}$ = Diſtance of the Center of Gravity of the Semi-ſegment $PADQ$ from QD.

Ex. 6.

163. *In the Parabola* $rx = yy$, let $SP = x$, $PM = y$, then in Reſpect of the Axis of Suſpenſion ST, $\dot{z} = yx\dot{x} = x\dot{x}\sqrt{rx}$; and $z = \frac{2}{5}xx\sqrt{rx}$: And $s = \frac{2}{3}xy = \frac{2}{3}x\sqrt{rx}$;

therefore $\frac{z}{s} = \frac{3}{5}x$ the Diſtance of the Center of Gravity from *ST*.

Again in Regard to the Axis of Suſpenſion *SP*, becauſe the Center of Gratity of the deſcribing Line is in the Middle of *MP*, therefore $\dot{z} = \frac{1}{2}yy\dot{x} = \frac{y^3\dot{y}}{r}$, and thence $z = \frac{y^4}{4r}$; and $s = \frac{2y^3}{3r}$: Therefore $\frac{z}{s} = \frac{3}{8}y$, the Diſtance from *SP*.

Ex. 7:

164. *For the hyperbolic Area BCMP, between the Aſſymptotes.* Let $SB = b$, $BC = c$, $SP = x$, $PM = y$, $cb = xy$. Then in Regard to the Axis *SD*, $\dot{z} = yx\dot{x} = cb\dot{x}$, and $z = cbx$, but in *B*, $x = b$, therefore by Correction $z = cb \times \overline{x - b}$. And $\frac{z}{s} = \frac{cb \times \overline{x-b}}{\text{Area } BCMP}$.

Again for the Axis of Suſpenſion *SP*, $\dot{z} = \frac{1}{2}yy\dot{x} = \frac{ccbb}{2xx}\dot{x}$, and $z = \frac{-ccbb}{2x} = \frac{-cby}{2}$; corrected $z = \frac{cb}{2} \times \overline{c-y}$ and $\frac{z}{s} = \frac{cb \times \overline{c-y}}{2 \text{ Area } BCMP}$.

Ex. 8.

165. *Let AMB be an Ellipſis,* *S* the Center, $AS = a$, $SB = b$, $SQ = y$, $QM = x = \frac{a}{b}\sqrt{bb-yy}$, then for the elliptic Space, $\dot{z} = xy\dot{y} = \frac{ay\dot{y}}{b}\sqrt{bb-yy}$; and $z = -\frac{a}{3b} \times \overline{bb-yy}^{\frac{3}{2}}$, corrected $z = \frac{abb}{3} - \frac{a}{3b} \times \overline{bb-yy}^{\frac{3}{2}}$: Thence $\frac{z}{s} = \frac{ab^3 - a \times \overline{bb-yy}^{\frac{3}{2}}}{3b \times \text{Area } SAMQ}$.

Likewiſe for the Diſtance from *SB*, $\dot{z} = \frac{1}{2}xx\dot{y} = \frac{aa\dot{y}}{2} - \frac{aay^2\dot{y}}{2bb}$, and $z = \frac{aa\dot{y}}{2} - \frac{aay^3}{6bb}$; whence $\frac{z}{s} = \frac{3bbaay - aay^3}{6bb \times \text{Area } SAMQ}$. FIG.

Ex. 9.

Let SMP be the hyperbolic Space, Tranſverſe $= 2a$, Conjugate $= 2b$, $SP = x$, $PM = y = \frac{b}{a}\sqrt{2ax + xx}$, whence $xx = 2aa + \frac{aa}{bb}yy - \frac{2aa}{b}\sqrt{bb + yy}$, $x\dot{x} = \frac{aa}{bb}y\dot{y} - \frac{aay\dot{y}}{b\sqrt{bb + yy}}$; whence $\dot{z} = yx\dot{x} = \frac{aa}{bb}y^2\dot{y} - \frac{aay^2\dot{y}}{b\sqrt{bb + yy}}$, and $z = \frac{aay^3}{3bb} - \frac{aay}{2b}\sqrt{bb + yy} + \frac{aab}{2} \times 2.30258 \text{ Log. } y + \sqrt{bb + yy}$, whence $\frac{z}{s} = \frac{z}{\text{Area } SMP} =$ Diſtance from *ST*. 163.

Then for the Diſtance from *SP*, we have $\dot{z} = \frac{yy\dot{x}}{2} = \frac{bb\dot{x}}{2aa} \times \overline{2ax + xx}$, and the Fluent $z = \frac{bbxx}{2a} + \frac{bbx^3}{6aa}$; therefore $\frac{z}{s} = \frac{3abbxx + bbx^3}{6aa \times \text{Area } SPM}$.

Ex. 10.

For the Surface of a right Cone, let $SD = f$, $c =$ ircumference of the Baſe, Axis $SB = d$, $SM = v$, $F = x$. Then it is plain its Center of Gravity is in he Axis *SB*. $\frac{cx}{d} =$ Circumference of the Circle *Q*; and by ſimilar Triangles $v = \frac{fx}{d}$, and $\dot{v} =$ 166.

$\frac{f\dot{x}}{d}$; therefore $\dot{z} = \frac{cx^2\dot{v}}{d} = \frac{cfx^2\dot{x}}{dd}$; and $z = \frac{cfx^3}{3dd}$; alſo $s = \frac{cxv}{2d} = \frac{cfxx}{2dd}$: Therefore $\frac{z}{s} = \frac{2}{3}x = SC$.

Ex. 11.

166. *For the Cone* (*or Pyramid*) *SDE*, let the Baſe $= b$, the reſt as in the laſt Example; then the Circle MQ $= \frac{bxx}{dd}$; and $\dot{z} = \frac{bx^3\dot{x}}{dd}$, and $z = \frac{bx^4}{4dd}$; alſo s $= \frac{bx^3}{3dd}$: Therefore $\frac{z}{s} = \frac{3}{4}x = SC$ the Diſtance of the Center of Gravity from S.

Ex. 12.

167. *Let SMD be a Sphere*, $SP = x$, $PM = y$, Radius $= r$, $c = 3.1416$, then $y = \sqrt{2rx - xx}$, and $\dot{z} = cy^2x\dot{x}$ $= 2crxx\dot{x} - cx^3\dot{x}$; therefore $z = \frac{2crx^3}{3} - \frac{cx^4}{4}$; and $s = crx^2 - \frac{cx^3}{3}$ (by Ex. 3. Prob. XIV.); therefore $\frac{z}{s} = \frac{8rx - 3xx}{12r - 4x}$ for the Diſtance of the Center of Gravity from S.

Ex. 13.

167. *For the Spheroid SMD, whoſe Center is C*, let $SC =$ a, $CF = b$, $SP = x$, $PM = y$, $c = 3.1416$, then $yy =$ $\frac{bb}{aa} \times \overline{2ax - xx}$; and $\dot{z} = cyyx\dot{x} = \frac{cbb}{aa} \times \overline{2ax^2x - x^3\dot{x}}$ and $z = \frac{cbb}{aa} \times \overline{\frac{2}{3}ax^3 - \frac{1}{4}x^4}$; and $s = \frac{cbb}{aa} \times \overline{ax^2 - \frac{1}{3}x^3}$ and $\frac{z}{s} = \frac{8ax - 3xx}{12a - 4x}$.

Ex

Ex. 14.

To find the Center of Gravity of the Solid SBDm, generated by a partial Revolution of the Parabola SMD about the Axis SB. FIG. 168.

Let S be the Point of Suſpenſion, let $SB = d$, $BD = b$, $SP = x$, $PM = y$, $ax = yy$, Arch $Dd = c$; then Arch $Mm = \frac{cy}{b}$; therefore $\dot{z} = \frac{cyy}{2b} x\dot{x} = \frac{cax^2\dot{x}}{2b}$; and $z = \frac{cax^3}{6b}$: and $s = \frac{cax^2}{4b}$, therefore $\frac{z}{s} = \frac{2}{3}x$, the Diſtance from ST.

Again for its Diſtance from SB, let Chord $Dd = f$, then Chord $Mm = \frac{fy}{b}$; and by Ex. 4. the Diſtance of the Center of Gravity of the Sector PMm from $P = \frac{2fy}{3c}$; therefore $\dot{z} = \frac{fy^3\dot{x}}{3b} = \frac{2fy^4\dot{y}}{3ab}$; and $z = \frac{2fy^5}{15ab}$; therefore $\frac{z}{s} = \frac{8fy^5}{15caax^2} = \frac{8fy}{15c}$, the Diſtance of the Center of Gravity from SP, and in the Plane that paſſes through the Axis and biſſects the Baſe.

Ex. 15.

Let the Hyperbola CM revolve round the Aſymptote SP, and deſcribe an Hyperboloid CMB: Let $SB = b$, $BC = d$, $SP = x$, $PM = y$, $c = 3,1416$. $bd = xy$; and $\dot{z} = cyyx\dot{x} = \frac{bbddc\dot{x}}{x}$, whence $z = bbddc$ Log. x: And corrected $z = bbddc$ Log. $\frac{x}{b}$: Alſo $\dot{s} = cyy\dot{x} = \frac{cbbdd\dot{x}}{xx}$; and $s = -\frac{cbbdd}{x}$, and corrected $s = cbbdd \times \frac{x-b}{bx}$; therefore $\frac{z}{s} = \frac{bx}{x-b} \times$ Log. $\frac{x}{b}$ 169.

FIG. = Diſtance of the Center of Gravity of the Solid from *SA*.

Ex. 16.

163. *Let the Solid be an Hyperboloid,* Tranſverſe $= 2a$, Conjugate $= 2b$, $c = 3.1416$, $SP = x$, $PM = y$, $= \frac{b}{a}\sqrt{2ax+xx}$; whence $\dot{z} = cyyx\dot{x} = \frac{cbb\dot{x}}{aa} \times \overline{2ax^2+x^3}$; whence $z = \frac{cbb}{aa} \times \overline{\frac{2}{3}ax^3+\frac{1}{4}x^4}$; alſo $\dot{s} = \frac{cbb}{aa}\dot{x} \times \overline{2ax+xx}$; and $s = \frac{cbb}{aa} \times \overline{axx+\frac{1}{3}x^3}$; therefore $\frac{z}{s} = \frac{8ax+3xx}{12a+4x}$ = Diſtance of the Center of Gravity from the Vertex *S*.

PROB. XVIII.

To find the Centers of Percuſſion aud Oſcillation.

The Center of Percuſſion is that Point in the Axis of a vibrating Body, which ſtriking againſt an immovable Obſtacle, the Body ſhall incline to neither Side, but reſt as it were in Equilibrio, on that Point.

And the Center of Oſcillation is the Point in the Axis of a vibrating Body, in which if a ſmall Body or Particle be placed, it ſhall perform its Vibrations after the ſame Manner, in the ſame Time, and with the ſame angular Velocity as the whole Body.

170. To find the Center of Percuſſion; through the Point of Suſpenſion *C*, and Center of Gravity, draw the Axis of the Body *CO*; and ſuppoſe *O* to be the Center of Percuſſion; through *CO* draw the Plane in which the Center of Gravity moves, and imagine the Body to be divided into innumerable ſmall Priſms,

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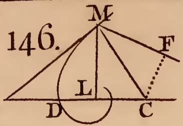

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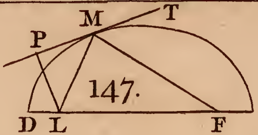

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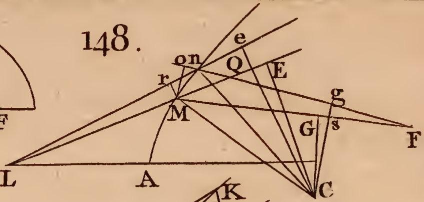

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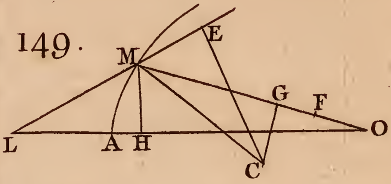

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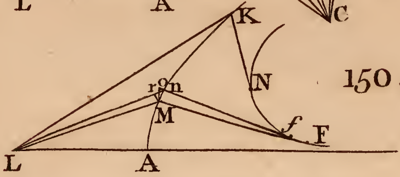

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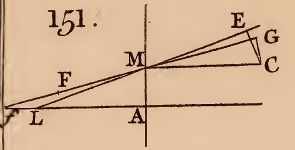

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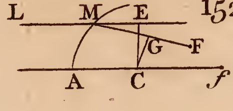

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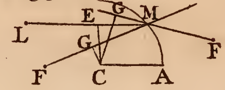

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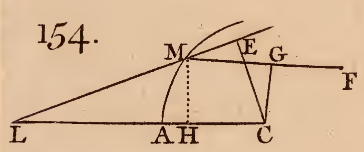

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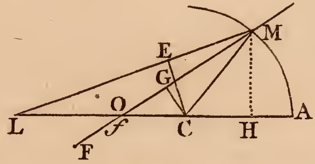

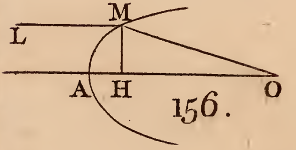

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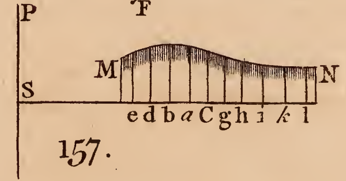

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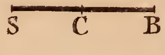

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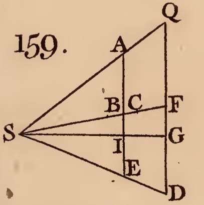

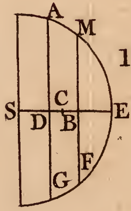

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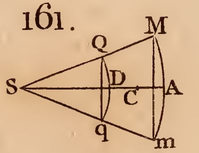

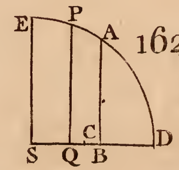

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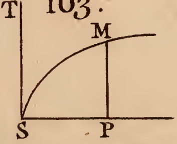

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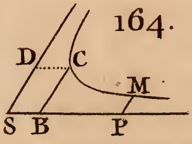

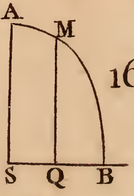

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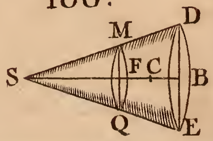

Pl. VII. p. 308.

all perpendicular to this Plane, and let them be ſuppoſed to be reduced to, or ſituated in, the Points where they interſect this Plane; and let p be one of theſe ſmall Priſms. Draw pf perpendicular to CO, and pd perpendicular to Cp; then pd will be the Direction of p's Motion as it revolves about C; and the Body being ſtopt at O, p will urge the Point d forward, with a Force proportional to its Magnitude and Velocity, that is as $p\times Cp$; therefore the Force wherewith p acts at d in a Direction perpendicular to CO, will be $p\times Cf$. And the Force by which p endeavours to turn the Body about O, will be as $p\times Cf\times do$, or $p\times Cf\times\overline{CO-Cd}$, that is as $p\times Cf\times CO-p\times Cp^2$. Now ſince the Sum of all theſe Forces to turn the Body about O muſt be $=0$, therefore all the $p\times Cf\times CO-p\times Cp^2=0$, or all the $p\times Cf\times CO=$ all the $p\times Cp^2$; therefore $CO=\dfrac{\text{Sum of all the } p\times Cp^2}{\text{Sum of all the } p\times cf}$.

171. For the Center of Oſcillation. Through the Centers of Motion C and of Gravity G draw the Axis CO, and let O be the Center of Oſcillation. Draw the horizontal Line Cr, and Or, Gg, pn perpendicular thereto, and pf perpendicular to CO.

By Reaſon of the equal angular Velocities of all the Particles of the Body; the abſolute Motion of any Particle q (and conſequently the Force that generates it) will be as $Cq\times q$; and therefore a Force acting at n that can generate that Motion in q, is as $\dfrac{Cq}{Cn}\times Cq\times q$, or $\dfrac{Cq^2\times q}{Cn}$, by the Power of the Leaver: Let $s=$ Sum of all the $Cq^2\times q$ in the Body, and let this be as (the Weight of) the Particle p. If the Weight of the Particle p, acting at n, generates any Motion in the whole Body AD, then $\dfrac{Cq^2\times q}{s}p=$ that Part of the Gravity of p which generates the Motion of the

FIG. 171. the Particle q; and the Force acting at q, and the Motion of q generated by that Force, is $\frac{Cn}{Cq} \times \frac{Cq^2 \times q}{s}p$ or $\frac{Cn \times Cq \times q}{s}p$ and the Velocity of q is $\frac{Cn \times Cq \times p}{s}$, and its angular Velocity $\frac{Cn \times p}{s}$. After the ſame Manner any other Force $\frac{Cq^2 \times q}{s}p$ acting at n, will generate the ſame angular Velocity in any other Particle q; and conſequently the Sum of all the Forces $\frac{Cq^2 \times q}{s}p$ or the Weight p will generate the ſame angular Velocity in all the Particles q together or in the whole Body. Now ſince the Weight of any Particle p will generate an angular Velocity in the Body AD about C, which is as $\frac{Cn \times p}{\text{Sum of all the } Cp^2 \times p}$; therefore the angular Velocity which all the Particles p can generate is as $\frac{\text{the Sum of all the } Cn \times p}{\text{Sum of all the } Cp^2 \times p}$. In like Manner the angular Velocity which the Gravity of a Particle p placed in O would generate in itſelf is as $\frac{Cr \times p}{CO^2 \times p}$, or as $\frac{Cr}{CO}$. But becauſe of the Equality of the Vibrations and correſpondent Accelerations, this laſt muſt be equal to the Sum of the former; whence $\frac{\text{Sum of } Cn \times p}{\text{Sum of } Cp^2 \times p} = \frac{Cr}{CO^2}$: But by the Nature of the Center of Gravity, the Sum of all the $Cn \times p = Cg \times$ Body $AD = \frac{Cg}{CG} \times CG \times$ Body AD $= \frac{Cr}{CO} \times$ Sum of all the $Cf \times p$. Therefore $\frac{Cr}{CO} \times \frac{\text{Sum } Cf \times p}{\text{Sum } Cp^2 \times p} = \frac{Cr}{CO^2}$; whence $CO =$

Sum

FIG. 171.

$\frac{\text{Sum of all } Cp^2\times p}{\text{Sum of all } Cf\times p}$: And therefore the Center of Oſcillation is the ſame with the Center of Percuſſion.

Since Sum of $Cf\times p = CG\times$ Body AD; therefore $CO = \frac{\text{Sum of } Cp^2\times p}{CG\times \text{Body } AD}$.

Alſo ſince $Cp^2 = CG^2 + Gp^2 - 2CG\times Gf$, therefore the Sum of all $Cp^2\times p$ = Sum of all $\overline{GC^2+Gp^2}\times p$ − Sum of all $2CG\times Gf\times p$. But by the Nature of the Center of Gravity, the Sum of all the $Gf\times p = 0$; therefore Sum $Cp^2\times p$ = Sum $CG^2\times p$ + Sum $Gp^2\times p = CG^2 \times$ Body AD + Sum $Gp^2\times p$. Therefore $CO = \frac{\text{Sum } Cp^2\times p}{CG\times \text{Body } AD} = CG + \frac{\text{Sum of } Gp^2\times p}{CG\times \text{Body } AD}$. And $GO = \frac{\text{Sum of all } Gp^2\times p}{CG\times \text{Body } AD}$.

Therefore if s = Body, $Cp = x$, $Cf = v$, $Gp = z$, then the Sum of all the $Cp^2\times p$ = Sum of $x^2\acute{s}$ = Fluent of $x^2\dot{s}$, and Sum of $Cf\times p$ = Sum of $v\acute{s}$ = Fluent of $v\dot{s}$. And Sum of $Gp^2\times p$ = Sum $z^2\acute{s}$ = Fluent of $z^2\dot{s}$; therefore $CO = \frac{\text{Fluent of } x^2\dot{s}}{\text{Fluent of } v\dot{s}} = \frac{\text{Fluent of } x^2\dot{s}}{CG\times \text{Body } AD} = CG + \frac{\text{Fluent of } z^2\dot{s}}{CG\times \text{Body } AD}$. Whence this

RULE.

Draw a parallel Line to the Axis of Motion, through the Center of Gravity of the Figure (whether it be Line, Surface or Solid); and find the Sum of all the Products of every Particle of the Figure, multiply'd by the Square of its Diſtance from the Axis of Motion, or elſe from the parallel Line; which is done thus,

1. *Multiply the (Moment or) Fluxion of the Figure, by the Square of its Diſtance from the Axis of Motion, and*

FIG. 171. *and (by Help of the Equation of the Curve) find the Fluent F; or elſe multiply by the Square of its Diſtance from the parallel Line, and find the Fluent G.*

But in many Caſes, eſpecially of Solids, the Fluent cannot be had at one Operation; and then you muſt firſt find the Sum of theſe Products in the generating Line, or Plane, of the Figure propoſed; thus. Multiply the (Moment or) Fluxion of that generating Line or Plane, by the Square of its Diſtance from the Axis of Motion, or elſe from the parallel Line; and find the Fluent. And then multiply this by the Fluxion of the Abſciſſa of the Figure; and find the Fluent F, or elſe G.

2. *Then draw a Line thro' the Point of Suſpenſion and the Center of Gravity; and from the generating Point, Line, or Plane of the Figure, let fall a Perpendicular upon it; and find the Diſtance of that Perpendicular from the Point of Suſpenſion; then multiply the Fluxion of the Figure by that Diſtance, and (from the Equation of the Figure) find the Fluent M. And let* $d =$ *Diſtance from the Point of Suſpenſion to the Center of Gravity, and* $B =$ *Body or Figure given. Then* $\frac{F}{M}$ *or* $\frac{F}{dB}$ *or* $d + \frac{G}{dB}$ *will be the Diſtance of the Center of Percuſſion or Oſcillation from the Point of Suſpenſion.*

SCHOLIUM.

If the Center of Percuſſion or Oſcillation be made the Center of Suſpenſion, then the former Point or Center of Suſpenſion, becomes the Center of Percuſſion; if the Plane of its Motion remains the ſame. And in general, the Diſtance of the Center of Suſpenſion from the Center of Percuſſion or Oſcillation, in the ſame Body, will always remain the ſame; if the Diſtance of its Center of Gravity from the Point of Suſpenſion, and the Plane of its Motion (in regard to the Body) remain the ſame. For then *d* and

G

G remain the ſame, and $\frac{G}{dB}$ = Diſtance of the Centers of Gravity and Percuſſion. And therefore in this Caſe if the Body be put into any oblique Poſition, it makes no Difference. Likewiſe if the Diſtances of the Center of Gravity from the Centers of Suſpenſion and Oſcillation, be known in one Caſe, they are known in all Caſes, the Plane of the Motion remaining the ſame. For their Rectangle $= \frac{G}{B}$; and therefore they are reciprocally proportional. And hence, as to the Calculation, it will be ſufficient to find the Center of Oſcillation in the ſimpleſt Caſe, and then it may be eaſily had in any other, or when the Point of Suſpenſion is at any other Diſtance aſſigned.

Cor. The Center of Preſſure of any Plane immerſed 171.
in a Fluid, and ſuſtaining that Fluid, is the ſame with the Center of Percuſſion of that Plane; the Axis of Motion being the Interſection of this Plane with the Surface of the Fluid. The Center of Preſſure is that Point againſt which a Force being applied = Sum of all the Preſſures, ſhall juſt ſuſtain them; ſo as the Plane ſhall incline to neither Side.

Through the Center of Gravity of the Plane draw 172.
AO perpendicular to AS the Interſection of the Plane and Surface of the Fluid, and let cd be parallel to AS. Then the Preſſure againſt any ſmall Part cd is as $cd \times Ab$, and its Force to turn the Plane about O the Center of Preſſure is $cd \times Ab \times bO = cd \times Ab \times AO - cd \times Ab^2$, and the Sum of all theſe muſt be equal to 0, therefore $AO = \frac{\text{Sum of } cd \times Ab^2}{\text{Sum of } cd \times Ab}$, therefore O is the ſame with the Center of Oſcillation and Percuſſion, and conſequently is to be found the ſame Way.

EXAMPLE I.

FIG. 173. *Let CB be a right Line*, $CB = x$, then $\dot{F} = x^2\dot{x}$ and $F = \frac{x^3}{3}$; alſo $\dot{M} = x\dot{x}$, and $M = \frac{x^2}{2}$; whence $\frac{F}{M} = \frac{2}{3}x = CO$.

Ex. 2.

174. *In a Parallelogram* where the Axis of Motion is in the Plane of the Figure, $CB = x$, $BD = b$, then $\dot{F} = bx^2\dot{x}$, and $F = \frac{bx^3}{3}$; alſo $\dot{M} = bx\dot{x}$, and $M = \frac{bx^2}{2}$; whence $\frac{F}{M} = \frac{2}{3}x = CO$.

Ex. 3.

175. *Let ABD be the Arch of a Circle*, the Center C the Point of Suſpenſion, and the Axis of Motion perpendicular to its Plane; let Arch $ABD = s$, Cord $AD = c$, $CB = r$. Then $F = rrs$; and by Ex. 3. Prob. XVII. $d = \frac{rc}{s}$; therefore $\frac{F}{dB} = \frac{F}{ds} = \frac{rs}{c}$.

Ex. 4.

176. *Let AD be a right Line*, the Axis of Motion perpendicular to the Plane paſſing through it. $CB = d$, $BA = y$, then $\dot{F} = \overline{dd + yy} \times 2\dot{y}$, and $F = 2ddy + \frac{2}{3}y^3$; alſo $M = 2dy$; whence $\frac{F}{M} = \frac{2ddy + \frac{2}{3}y^3}{2dy} = d + \frac{yy}{3d}$.

Ex. 5.

177. *For the Periphery of a Circle*, let $CD = d$, Radius $AD = r$, Circumference $= c$. If the Axis of Motion be perpendicular to its Plane, then $G = rrc$, and $d + \frac{G}{dB} = d + \frac{rrc}{dc} = d + \frac{rr}{d}$.

But if the Axis of Motion be in the Plane of the Circle, let $DE = z$, $PQ = s$, then $\dot{G} = z^2\dot{s} = \frac{4rz^2\dot{z}}{\sqrt{rr-zz}}$, and by Form 17, the Fluent $G = \frac{rrc}{2}$, for the whole Circle: Therefore $d + \frac{G}{dB} = d + \frac{rr}{2d}$. FIG.

Ex. 6.

For the Plane of the Circle, the Axis of Motion perpendicular to its Plane, $DE = z$, Circumference at $E = \frac{cz}{r}$, then $\dot{G} = \frac{cz^3\dot{z}}{r}$, and $G = \frac{cz^4}{4r} = \frac{cr^3}{4}$. Then $d + \frac{G}{dB} = d + \frac{cr^3}{4dB} = d + \frac{rr}{2d}$. 177.

And if the Axis of Motion be in the Plane of it, $\dot{G} = 4z^2\dot{z}\sqrt{rr-zz}$, whence $G = \frac{cr^3}{8}$; and $d + \frac{G}{dB} = d + \frac{cr^3}{8dB} = d + \frac{rr}{4d}$.

Ex. 7.

For the Periphery of the Circle, parallel to the Horizon, let the Axis of Motion be parallel to *ED*, Radius $DB = r$, $BA = v$, $DL = z$, $CD = d$, then $\dot{G} = 4zz\dot{v} = \frac{4rzz\dot{z}}{\sqrt{rr-zz}}$, and (by Form 17) the whole Fluent $G = \frac{rrc}{2}$; then $d + \frac{G}{dB} = d + \frac{rrc}{2dc} = d + \frac{rr}{2d}$. 178.

Ex. 8.

For the Plane of the Circle, whoſe Point of Suſpenſion is in *CD* perpendicular to its Plane. Let $AL = x$, $DL = z$, then $\dot{G} = 4xz^2\dot{z} = 4z^2\dot{z} \times \sqrt{rr-zz}$, and 178.

and the whole Fluent $G = \frac{cr^3}{8}$; and $d + \frac{G}{dB} = d + \frac{rr}{4d}$.

Ex. 9.

179. *In an Iſoceles Triangle* where the Axis of Motion is parallel to the Baſe, let $CD = a$, $CA = x$, $BE = f$, then $IK = \frac{fx}{a}$, and $\dot{F} = \frac{fx^3\dot{x}}{a}$, whence $F = \frac{fx^4}{4a}$; alſo $\dot{M} = \frac{fx^2\dot{x}}{a}$, and $M = \frac{fx^3}{3a}$; therefore $\frac{F}{M} = \frac{3}{4}x = \frac{3}{4}a$.

If the Axis at C be perpendicular to the Plane of the Triangle, let $AQ = v$, then $CQ^2 \times$ Fluxion of $AQ = \overline{xx+vv} \times 2\dot{v}$, and x being given, the Fluent $= 2x^2v + \frac{2v^3}{3}$, then $\dot{F} = 2x^2v\dot{x} + \frac{2v^3\dot{x}}{3} =$ (becauſe $v = \frac{fx}{2a}$) $\frac{fx^3\dot{x}}{a} + \frac{f^3x^3\dot{x}}{12a^3}$; whence $F = \frac{fx^4}{4a} + \frac{f^3x^4}{48a^3} = \frac{12fa^3+f^3a}{48}$; therefore $\frac{F}{M} = \frac{12aa+ff}{16a}$.

Ex. 10.

180. *In the Parabola* CAF, $AC = x$, $AF = y$, $AQ = v$, $ax = yy$. Let the Axis in C be parallel to AF, then $\dot{F} = yx^2\dot{x}, = x^2\dot{x}\sqrt{ax}$, and $F = \frac{2}{7}x^3\sqrt{ax}$; and $\dot{M} = yx\dot{x} = x\dot{x}\sqrt{ax}$, and $M = \frac{2}{5}x^2\sqrt{ax}$. Therefore $\frac{F}{M} = \frac{5}{7}x$.

If the Axis be perpendicular to its Plane, then $CQ^2 \times AQ = \overline{xx+vv} \times \dot{v}$, and the Fluent $= x^2v + \frac{1}{3}v^2$, x being given; then $\dot{F} = x^2v\dot{x} + \frac{v^3\dot{x}}{3} = yx^2\dot{x} + \frac{y^3\dot{x}}{3} = x^2\dot{x}\sqrt{ax} + \frac{ax\dot{x}\sqrt{ax}}{3}$; whence $F = \frac{2}{7}x^3\sqrt{ax} +$

$\frac{2}{15}ax^2\sqrt{ax}$. Therefore $\frac{F}{M} = \frac{5}{7}x + \frac{1}{3}a$.

Ex. 11.

Let AB be the Surface of a Sphere, $AB = s$, $BE = z$, Radius $AD = r$, c = Circumference. Then Circumference of $BE = \frac{cz}{r}$, and $\dot{G} = \frac{cz^3\dot{s}}{r} = \frac{cz^3}{r} \times \frac{r\dot{z}}{\sqrt{rr-zz}}$; whence $G = \frac{2cr^3}{3} - \overline{\frac{2cr^2}{c} - \frac{czz}{3}}\sqrt{rr-zz} = \frac{2cr^3}{3}$. And $B = rc$; therefore $d + \frac{G}{dB} = d + \frac{2rr}{3d}$. 181.

Ex. 12.

Let ABQD be a Parallelepipidon, the Axis of Motion perpendicular to the Plane $ABQD$; let $AB = 2a$, $AD = 2b$, Breadth $= c$, $GS = x$, $SZ = y$; then $GZ^2 \times \dot{ASz} = \overline{xx+yy} \times 4c\dot{y}$, and the Fluent $= 4cx^2y + \frac{4}{3}cy^3$, x being given; whence $\dot{G} = 4cx^2y\dot{x} + \frac{4cy^3\dot{x}}{3} = 4cbx^2\dot{x} + \frac{4}{3}cb^3\dot{x}$; therefore $G = \frac{4}{3}cbx^3 + \frac{4}{3}cb^3x = \frac{4}{3}cba^3 + \frac{4}{3}cb^3a$; alſo $B = 4acb$: Therefore $d + \frac{G}{dB} = d + \frac{aa+bb}{3d}$. 182.

Ex. 13.

In a Cylinder let $CA = x$, $AD = r$, $Ae = y$, $CH = a$, c = Circumference, the Axis of Motion parallel to AB; then for the Circle ADB, $Ce^2 \times De \times \dot{Ae} = \overline{xx+yy} \times 4\dot{y}\sqrt{rr-yy}$, and the whole Fluent $= \frac{rcx^2}{2} + \frac{r^3c}{8}$. Then $\dot{F} = \frac{rcx^2\dot{x}}{2} + \frac{r^3c\dot{x}}{8}$, and $F = \frac{rcx^3}{6} + \frac{r^3cx}{8}$; which corrected gives $F = \frac{rcx^3 - rca^3}{6}$ 183.

+

$+\dfrac{r^3cx-r^3ca}{8}$; and $B=\overline{x-a}\times\dfrac{rc}{2}$, $d=\dfrac{x+a}{2}$, then will $\dfrac{F}{dB}=\dfrac{2x^3-2a^3}{3x^2-3aa}+\dfrac{rrx-rra}{2x^2-2aa}=\dfrac{4xx+4ax+4aa+3rr}{6x+6a}$.

Ex. 14.

184. *Let CKB be a Pyramid*, whoſe Baſe is a Parallelogram, and Axis of Motion in C parallel to the Side DA. Let its Altitude $=a$, $AB=f$, $AD=c$, $CH=x$, $HL=y$, then $KI=\dfrac{cx}{a}$, and $EF=\dfrac{fx}{a}$, and $CL^2\times KI\times\dot{\overline{HL}}=\overline{xx+yy}\times\dfrac{cx\dot{y}}{a}$, and the Fluent $=\dfrac{cx^3y}{a}+\dfrac{cy^3x}{3a}$; whence $\dot{F}=\dfrac{cx^3\dot{x}}{a}y+\dfrac{cx\dot{x}y^3}{3a}=\dfrac{cfx^4\dot{x}}{aa}+\dfrac{cf^3x^4\dot{x}}{12a^4}$. And $F=\dfrac{cfx^5}{5aa}+\dfrac{cf^3x^5}{60a^4}$. Alſo $\dot{M}=\dfrac{cfx^3\dot{x}}{aa}$, and $M=\dfrac{cfx^4}{4aa}$; whence $\dfrac{F}{M}=\frac{4}{5}x+\dfrac{ff}{15aa}x=\dfrac{12aa+ff}{15a}$.

Ex. 15.

185. *In a right Cone*, let $CA=g$, Altitude $=a$, Radius of the Baſe $=f$, $AB=x$, $BI=z$, $c=3{,}1416$ then BE or $BD=\dfrac{fx}{a}$, $IE=\sqrt{\dfrac{ffxx}{aa}-zz}$; then for the Plane DEB, $CI^2\times IE\times 4\dot{\overline{BI}}=\overline{g+x}^2+zz:\times 4\dot{z}\sqrt{\dfrac{ffxx}{aa}-zz}$, and the Fluent $=\overline{g+x}^2\times\dfrac{cffx^2}{aa}+\dfrac{cf^4x^4}{4a^4}$, x being given. Then $\dot{F}=\overline{g+x}^2\times$

$cffx^2\dot{x}$

$\frac{cffx^2\dot{x}}{aa} + \frac{cf^4x^4\dot{x}}{4a^4}$, conſequently $F = : \frac{1}{3}ggcx^3 + \frac{1}{2}gcx^4 + \frac{1}{5}cx^5 + \frac{cffx^5}{20aa} : \times \frac{ff}{aa}$. Alſo $\dot{M} = \overline{g+x} \times \frac{cffx^2\dot{x}}{aa}$, and $M = : \frac{1}{3}gcx^3 + \frac{1}{4}cx^4 : \times \frac{ff}{aa}$. Whence $\frac{F}{M} = \frac{20gg + 30ga + 12aa + 3ff}{20g + 15a}$.

Ex. 16.

181. *For the Sphere AS.* Draw the Diameter *AS* parallel to the Axis of Motion at *C*. And let $AD = r$, $AE = v$, $Ef = z$, $EB = y$, $c = 3.1416$: Then for the Circle *BE*, $c \times 2Ef \times \dot{\overline{EF}} \times Ef^2 = 2cz^3\dot{z}$; and the Fluent $= \frac{1}{2}cz^4 = \frac{1}{2}cy^4$; whence $\dot{G} = \frac{1}{2}cy^4\dot{v} = \frac{1}{2}c\dot{v} \times \overline{2rv - vv}^2 = 2cr^2v^2\dot{v} - 2crv^3\dot{v} + \frac{1}{2}cv^4\dot{v}$; and $G = \frac{2}{3}cr^2v^3 - \frac{1}{2}crv^4 + \frac{1}{10}cv^5$. Alſo the Fluxion of the Solid $= cyy\dot{v} = c\dot{v} \times \overline{2rv - vv}$, and the Solid or the Segment $BAE = crv^2 - \frac{1}{3}cv^3 = B$. Then $\frac{G}{dB} = \frac{20r^2v - 15rv^2 + 3v^3}{d \times : 30r - 10v}$. And for the whole Sphere, $\frac{G}{dB} = \frac{2rr}{5d} = DO$; and $CO = d + \frac{2rr}{5d}$.

If v be very ſmall, then $DO = \frac{2rv}{3d} = \frac{yy}{3d}$, nearly; and in moſt Clocks the Bob of the Pendulum is in this Form.

Ex. 17.

186. *Let AD be a Paraboloid*; $CA = a$, $AB = x$, $BD = y$, $BI = z$, $c = 3.1416$, $rx = yy$. Then $EI \times \dot{\overline{BI}} \times 4CI^2 = \overline{a+x}^2 + zz : \times 4\dot{z}\sqrt{yy - zz}$, and the Fluent $= \overline{a+x}^2 \times cyy + \frac{1}{4}cy^4$; therefore $\dot{F} = \overline{a+x}^2 \times cyy\dot{x} + \frac{cy^4}{4}$

$\dot{x}$

FIG. $\dot{x} = \overline{a+x}^2 \times crx\dot{x} + \frac{1}{4}cr^2x^2\dot{x}$; whence $F = \frac{1}{2}caarxx + \frac{2}{3}carx^3 + \frac{1}{4}crx^4 + \frac{1}{12}crrx^3$. Alſo $\dot{M} = \overline{a+x} \times cy^2\dot{x} = \overline{a+x} \times crx\dot{x}$, and $M = \frac{1}{2}carx^2 + \frac{1}{3}crx^3$. Therefore $\frac{F}{M} = \frac{6a^2 + 8ax + rx + 3xx}{6a + 4x}$.

PROB. XIX.

187. *To find the Law of centripetal Force requiſite to cauſe a Body to move in a given Curve BF.*

Let B be the Place of the Body moving in the Orbit BF by a Force directed to the given Point C. Draw the Tangent BY, and the Radii CB, CQ, infinitely near each other. QR parallel to CB; and CY, Qn Perpendiculars to BY. Let the Diſtance $CB = D$, perpendicular $CY = P$, then the infinitely ſmall Line QR will be as the Force and Square of the Time conjunctly, that is as the Force and Square of the Area CBQ; therefore the Force is as $\frac{QR}{\overline{CBQ}^2}$ or $\frac{QR}{QB^2 \times P^2}$, that is (becauſe $P : D :: Qn : QR = \frac{Qn \times D}{P}$) as $\frac{Qn \times D}{QB^2 \times P^3}$. But $\frac{QB^2}{2Qn}$ = Radius of Curvature in the Point B, and the ſame Radius is alſo $\frac{D\dot{D}}{\dot{P}}$ by Prob. V. wherefore the Force is as $\frac{\dot{P}}{2P^3\dot{D}}$, that is, (ſuppoſing $\dot{D}$ to be given) as the Fluxion of $\frac{-1}{PP}$.

Therefore to find the Law of centripetal Force, let D = Diſtance from the Center of Force, P = perpendicular

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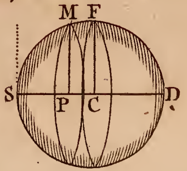

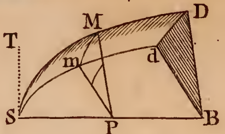

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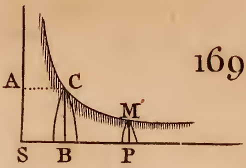

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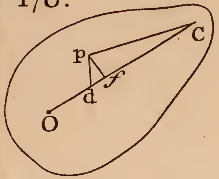

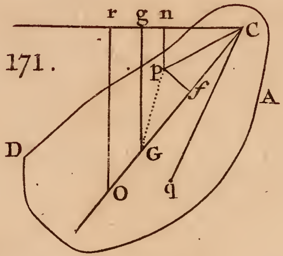

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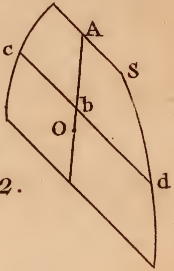

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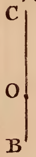

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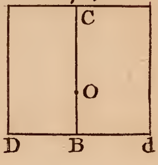

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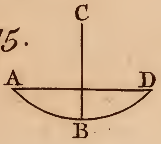

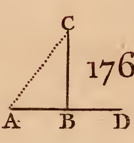

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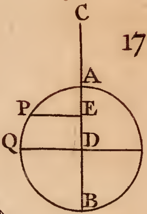

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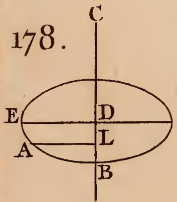

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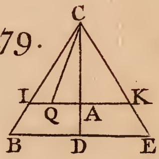

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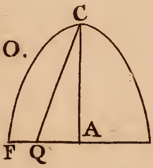

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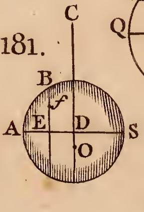

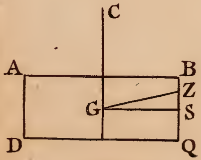

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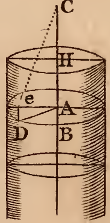

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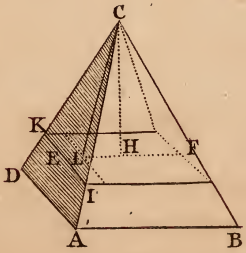

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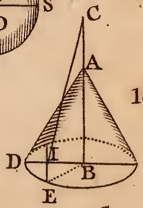

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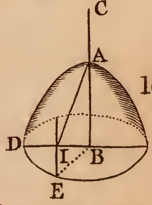


186.

Pl.VIII.p.320.

cular on the Tangent: Compute the Value of P in Terms of D, by the Nature of the Curve; then find the Fluxion of $\frac{-1}{PP}$, *making* $\dot{D}=1$, *and then expunge all Quantities as far as poſſible, except D, and you will have F, the Law of centripetal Force required.*

EXAMPLE 1.

Let C the Center of Force be in the Circumference of the Circle CBD; Diameter $CD=2r$, the Triangles *CBD* and *CBY* are ſimilar, whence $P=\frac{DD}{2r}$, therefore $\frac{-1}{PP}=\frac{-4rr}{D^4}$, therefore $F\propto\frac{16r^2\dot{D}}{D^5}\propto\frac{1}{D^5}$; that is, the Force is reciprocally as the Fifth Power of *D*. 188.

Ex. 2.

Let DB be a Circle; and C at an infinite Diſtance, $AE=x$, $ED=y$, $AD=r$, then $P=\frac{Dy}{r}$, and $\frac{-1}{PP}=\frac{-rr}{DD}y^{-2}$, whoſe Fluxion (becauſe *D* is a ſtanding Quantity) is $\frac{2r^2}{DD}y^{-3}\dot{y}$, therefore $F\propto\frac{2r^2}{D^2}y^{-3}$ or $F\propto\frac{1}{y^3}$. 189.

Ex. 3.

Let BF be an Ellipſis; *C* the Focus, *E* the Center; draw *BE* and its Conjugate *AE*, and let the tranſverſe Axis $=2r$, Conjugate $=2c$; then by the Property of the Ellipſis $2rD-DD$ the Rectangle of the focal Diſtances from *B*, is $=AE^2$, alſo *AE* or $\sqrt{2rD-DD}:c::D:P=\frac{cD}{\sqrt{2rD-DD}}$; and $\frac{-1}{PP}=\frac{-2rD+DD}{ccDD}$ = 190.

FIG. $= \frac{-2r}{ccD} + \frac{1}{cc}$, whoſe Fluxion is $\frac{2r\dot{D}}{ccDD}$, therefore $F \propto \frac{2r}{ccDD}$ or $\frac{1}{DD}$.

Ex. 4.

191. *Let C be the Center of the Ellipſis*; then by the Nature of the Figure $AC^2 + CB^2 = rr + cc$, and $AC = \sqrt{rr+cc-DD}$; alſo $AC : c :: r : P = \frac{cr}{\sqrt{rr+cc-DD}}$; therefore $-\frac{1}{PP} = \frac{-rr-cc+DD}{ccrr}$, whoſe Fluxion is $\frac{2D\dot{D}}{ccrr}$; therefore $F \propto \frac{2D}{ccrr} \propto D$. That is, the Force is as the Diſtance.

Ex. 5.

192. *Let BA be an Hyperbola, C* the Focus. Proceeding as in the Ellipſis, we ſhall find $P = \frac{cD}{\sqrt{2rD+DD}}$, and $-\frac{1}{P^2} = \frac{-2rD-DD}{ccDD} = -\frac{2r}{ccD} - \frac{1}{cc}$, whoſe Fluxion is $\frac{2r\dot{D}}{ccDD}$, therefore $F \propto \frac{2r}{ccDD}$ or $F \propto \frac{1}{DD}$.

And after the ſame Manner if the Force be in the other Focus, there will be found $P = \frac{cD}{\sqrt{DD-2rD}}$, and the Force $F \propto \frac{-2r}{ccDD} \propto \frac{-1}{DD}$, and is therefore a centrifugal Force,

Ex. 6.

193. *Let AB be an Hyperbola, C* the Center, Tranſverſe $=2r$, Conjugate $=2c$, $b =$ half the Conjugate belonging to *CB*. Then by the Property of the Hyperbola

$bb - DD = rr - cc$, and $b = \sqrt{rr - cc + DD}$; and $b : c :: r : P = \frac{cr}{\sqrt{rr - cc + DD}}$. Conſequently $-\frac{1}{PP} = \frac{-rr + cc - DD}{ccrr}$, whoſe Fluxion is $\frac{-2D\dot{D}}{ccrr}$, therefore $F \propto \frac{-2D}{ccrr} \propto -D$, and is therefore a centrifugal Force.

Ex. 7:

Let AB be a Parabola, C the Focus, r = Latus Rectum; by the Property of the Figure $P = \sqrt{\frac{rD}{4}}$, whence $-\frac{1}{PP} = \frac{-4}{rD}$, whoſe Fluxion is $\frac{4\dot{D}}{rDD}$; therefore $F \propto \frac{1}{DD}$. 194.

Ex. 8.

Let AB be a Parabola, C the Center of Force at an infinite Diſtance in the Axis. $AD = x$, $DB = y$, $ax = yy$, then $P = \frac{Dy}{\sqrt{4xx + ax}}$, and $\frac{-1}{PP} = \frac{-4xx - ax}{DDyy} = \frac{-4x}{aDD} - \frac{1}{DD}$, whoſe Fluxion is $\frac{-4\dot{x}}{aDD} + \frac{8x\dot{D}}{aD^3} + \frac{\dot{D}}{D^3} =$ (becauſe $-\dot{x} = \dot{D}$): $\frac{4D}{aD^3} + \frac{8x + a}{aD^3} : \times \dot{D}$; therefore $F \propto \frac{4D + 8x + a}{aD^3}$, or (becauſe D is infinite) $\propto \frac{4D}{aD^3}$, or $F \propto \frac{4}{aD^2}$ a given Quantity. 195.

Ex. 9.

Let C be the Vertex of the Parabola, $CA = x$, $AB = y$, $ax = yy$; by ſimilar Triangles $TB : AB :: TC$ or $x : CT$ or $P = \frac{xy}{\sqrt{4xx + yy}}$; whence $-\frac{1}{PP} = -$ 196.

FIG. $\frac{4xx+yy}{xxyy} = \frac{-4}{ax} - \frac{1}{xx}$; whoſe Fluxion is $\frac{4\dot{x}}{ax^2} + \frac{2\dot{x}}{x^3}$. But $DD = xx+ax$, and thence $\dot{x} = \frac{2D\dot{D}}{a+2x}$, therefore $F \propto \frac{4x+2a}{ax^3} \times \frac{2D\dot{D}}{2x+a} = \frac{4D\dot{D}}{ax^3}$, or $F \propto \frac{D}{x^3}$.

Ex. 10.

197. *Let VAR be an Ellipſis, C the Focus; and let the Curve VBQ be formed from the Ellipſis, thus; take CA $=Cb$, and the Angle VCA to the Angle VCb, as m to n, for all the Points of the Curve. To find the Law of centripetal Force of a Body moving in the Curve VBQ.*

Let the Tranſverſe of the Ellipſis $=2r$, Conjugate $=2c$. By the Property of the Ellipſis the Perpendicular $Cp = \frac{cD}{\sqrt{2rD-DD}}$, alſo $tb = TA$, and $\angle BCb = \frac{n}{m} ACo$. And by ſimilar Triangles $\frac{P \times tb}{BY} = tB = \frac{n}{m} TO = \frac{\frac{n}{m} Cp \times TA}{pA}$, that is $\frac{P}{\sqrt{DD-PP}} = \frac{nc}{m\sqrt{2rD-DD-cc}}$; whence by Reduction $PP = \frac{n^2c^2D^2}{2rm^2D - m^2D^2 - m^2c^2 + n^2c^2}$; whence $\frac{-1}{PP} = \frac{-2rm^2}{n^2c^2D} + \frac{m^2}{n^2c^2} - \frac{n^2c^2 - m^2c^2}{n^2c^2D^2}$; whoſe Fluxion is $\frac{2rm^2\dot{D}}{n^2c^2D^2} + \frac{2n^2-2m^2}{n^2c^2D^3}c^2\dot{D}$, therefore $F \propto \frac{m^2}{DD} + \frac{n^2-m^2}{rD^3}cc$. In the ſame Manner if C were the Center of the Ellipſis, it might be proved that the Force is as $\frac{m^2D}{rr} + \frac{nn-mm}{D^3}cc$.

Ex. 11.

Ex. 11.

FIG. 198. *Suppoſe AB to be the Logarithmic Spiral*; ſince the Angle *CBY* is always given, there is given the Ratio of *CB* to *CY* ſuppoſe as *m* to *n*. Then $P = \frac{n}{m} D$; and $\frac{-1}{PP} = \frac{-mm}{nnDD}$, whoſe Fluxion is $\frac{2m^2\dot{D}}{nnD^3}$; therefore $F \propto \frac{1}{D^3}$.

Ex. 12.

199. *Let CB be the hyperbolic Spiral*; draw *CT* perpendicular to *CB*, and let $CB = y$, the given Subtangent $CT = a$. By ſimilar Triangles $\sqrt{aa+yy} : y :: a : P = \frac{ay}{\sqrt{aa+yy}}$; and $-\frac{1}{PP} = -\frac{1}{yy} - \frac{1}{aa}$, whoſe Fluxion is $\frac{2\dot{y}}{y^3} = \frac{2\dot{D}}{D^3}$, therefore $F \propto \frac{1}{y^3} \propto \frac{1}{D^3}$.

PROB. XX.

200. *The Nature of the Curve ABD forming an Arch being given; to find the Nature of the Curve RST bounding the Top of the Wall ATRD ſupported by that Arch; by the Preſſure or Weight of which Wall, all the Parts of the Arch are kept in Equilibrio without falling.*

201. 1. Let ſeveral equal right Lines *AB*, *BC*, *CD*, *&c.* placed in a vertical Plane, be moveable round the Angles *A*, *B*, *C*, *D*, *&c.* whilſt the Points *A*, *G*, at the Baſe remain fixed and immoveable. Through *B*,

FIG. *B*, *C*, *D*, *&c.* draw the Lines *Bi*, *Cm*, *Dp*, *&c.* perpendicular to the Horizon; and complete the Parallelogram *Bhik*, and make $Cl = Bk$, and complete the Parallelogram *Clmn*. In like Manner make $Do = Cn$ or *lm*, $Er = op$, $Ft = rs$, and complete all the Parallelograms in the Figure as at firſt.

2. Let ſeveral Weights which are to one another as the Lines *Bi*, *Cm*, *Dp*, *&c.* lie reſpectively on the Points *B*, *C*, *D*, *&c.* Now the Force *Bi*, is equivalent to *Bh*, *Bk*, acting in the Directions *BA*, *BC*; the Force *Bh* is deſtroyed by the Reſiſtance of the Point *A*; but *Bk* endeavours to move the Point *B* towards *C*. In like Manner the Force *CM* is equivalent to *Cl* and *Cn*; the Force *Dp* to *Do*, *op*, *&c.* Now the Forces *Bk* (acting towards *C*) and *Cl* (acting towards *B*) being equal by Conſtruction deſtroy one another. In like Manner the Forces *Cn*, *Do*; *Dq*, and *Er*; *Ev* and *Ft*, *&c.* deſtroy one another; and the Point *G* being fixed, it is manifeſt the Figure *ABCD*, *&c.* will not be moved by the incumbent Weights, *Bi*, *Cm*, *Dp*, *&c.* but all its Parts will remain in Equilibrio.

3. The Force *Bh* : Force *Bk* or *Cl* : : Sine $\angle hiB$ or $iBC : S.\angle ABi :: \frac{1}{S.ABi} : \frac{1}{S.iBC}$ or $\frac{1}{S.mCB}$. Likewiſe Force *Cl* : Force *Cn* or *Do* : : $S.MCD$ or $pDC : S.mCB :: \frac{1}{S.mCB} : \frac{1}{S.mCD}$ or $\frac{1}{S.pDC}$, and ſo on; whence it is plain in general, that any Force *Cl* is as $\frac{1}{S.\angle mCB}$. Now ſince $Cm = \frac{S.Clm \times Cl}{S.Cml} = \frac{S.BCx \times Cl}{S.mCD}$; therefore the Force $Cm \propto \frac{S.BCx}{S.mCB \times S.mCD}$.

4. Now let the Number of the Lines *AB*, *BC*, *CD*, *&c.* be increaſed and their Lengths diminiſhed *ad infinitum*, that the Figure may obtain the Form of a Curve,

a Curve, and the Preſſure will then act on all Parts of it; and the Angle *BCx* will then become the Angle of Contact, and the Sines of *mCB* and *mCD* become equal to the Sine of *mCx*: Therefore drawing the Tangent *An* (Fig. 200) the Preſſure on any Point *A* to preſerve the Equilibrium will be as the Angle of Contact at *A* directly, and the Square of the Sine of the Angle *mAn* reciprocally. 200. But the Angle of Contact is as the Curvature, or reciprocally as the Radius of Curvature. Therefore the Preſſure is reciprocally as that Radius and the Square of the Sine of that Angle *mAn*.

5. Let $BC = x$, $AC = y$, $AB = z$. Radius of Curvature in $A = R$. Then if $\dot{z}$ be given, $S.\angle TAN \propto \dot{y}$, and $\frac{1}{S.TAN} \propto \frac{1}{\dot{y}}$. Then the Weight or Preſſure on $A = AT \times \dot{y}$, and that (as has been proved) is as $\frac{1}{R \times S.TAN^2} \propto \frac{1}{R\dot{y}^2}$, therefore $AT \propto \frac{1}{R\dot{y}^3}$; or in general $AT \propto \frac{\dot{z}^3}{R\dot{y}^3}$. But when $\dot{x}$ is given $R = \frac{\dot{z}^3}{-\dot{x}\ddot{y}}$; or if $\dot{y}$ be given $R = \frac{\dot{z}^3}{\dot{y}\ddot{x}}$; therefore $AT \propto \frac{-\dot{x}\ddot{y}}{\dot{y}^3}$ if $\dot{x}$ be given; or $AT \propto \frac{\ddot{x}}{\dot{y}^2}$, if $\dot{y}$ be given.

Wherefore to find the Curve ST, let $BC = x$, $AC = y$, $AB = z$; *then by the Nature of the Curve AB, compute* $\frac{-\dot{x}\ddot{y}}{\dot{y}^3}$ *if* $\dot{x}$ *be given, or* $\frac{\ddot{x}}{\dot{y}^2}$ *if* $\dot{y}$ *be given, and take AT proportional thereto. And x may be expunged out of the Value of AT, by Help of the given Line BS, and thence the Nature of the Curve ST will be known.*

EXAMPLE

EXAMPLE I.

FIG. 202. *Let BA be a Circle*, Radius $AR = r$, $BC = x$, $AC = y$, $BS = a$, $\dot{x}$ given. Then $y = \sqrt{2rx - xx}$, $\dot{y} = \frac{r - x}{\sqrt{2rx - xx}}\dot{x}$, $\ddot{y} = \frac{-rr\dot{x}^2}{\overline{2rx - xx}^{\frac{3}{2}}}$; then will $\frac{-\dot{x}\ddot{y}}{\dot{y}^3} = \frac{rr\dot{x}^3}{\overline{2rx - xx}^{\frac{3}{2}}} \times \frac{\overline{2rx - xx}^{\frac{3}{2}}}{\overline{r - x}^3 \times \dot{x}^3} = \frac{rr}{\overline{r - x}^3}$; therefore $AT \propto \frac{rr}{\overline{r - x}^3}$, or $AT = \frac{r^3 a}{\overline{r - x}^3}$. And expunging x, $SQ = (AT - CS =) \frac{r^3 a}{\overline{rr - yy}^{\frac{3}{2}}} - a - r + \sqrt{rr - yy}$, for the Nature of the Curve ST.

Hence the Curve ST runs upwards *ad infinitum*, and the Perpendicular DE is an Aſſymptote to the Curve.

SCHOLIUM.

If ST had been a right Line, then the Point A would be preſſed with too little Weight, and B with too much: And hence appears the Reaſon why circular Arches commonly break about the Top, by being loaded there with more Weight than their due Proportion.

Ex. 2.

202. *Let the Curve DAB be an Ellipſis*; $BR = r$, $DR = c$, $CB = x$, $AC = y$, $BS = a$, $\dot{x}$ given. Then $y = \frac{c}{r}\sqrt{2rx - xx}$, $\dot{y} = \frac{c\dot{x}}{r} \times \frac{r - x}{\sqrt{2rx - xx}}$, $\ddot{y} = \frac{-cr\dot{x}^2}{\overline{2rx - xx}^{\frac{3}{2}}}$: Whence $\frac{-\dot{x}\ddot{y}}{\dot{y}^3} = \frac{r^4}{cc \times \overline{r - x}^3}$, which is as AT; therefore

AT

FIG.

$AT = \frac{r^3a}{\overline{r-x}^3}$. Then $SQ = \frac{r^3a}{\overline{r-x}^3} - a - x$ $=$ (expunging x) $\frac{c^3a}{\overline{cc-yy}^{\frac{3}{2}}} - a - r + \frac{r}{c}\sqrt{cc-yy}$, for the Nature of the Curve ST, being of the ſame Kind with the foregoing.

Ex. 3.

Let AB be a Parabola, $BC=x$, $CA=y$, $SB=a$, 203.
$rx=yy$, $\dot{y}$ given. Then $\dot{x} = \frac{2y\dot{y}}{r}$, $\ddot{x} = \frac{2\dot{y}^2}{r}$, and $\frac{\ddot{x}}{\dot{y}^2} = \frac{2}{r}$, a given Quantity, therefore AT is every where the ſame, and ST is the ſame Parabola placed in a higher Poſition.

Ex. 4.

Let BA be an Hyperbola, Tranſverſe $= 2r$, Con- 203.
jugate $= 2c$, $BC=x$, $CA=y$, $BS=a$, $\dot{y}$ given. Then $2rx+xx = \frac{rryy}{cc}$, whence $r+x = \frac{r}{c}\sqrt{cc+yy}$, $\dot{x} = \frac{r\dot{y}}{c} \times \frac{y}{\sqrt{cc+yy}}$, $\ddot{x} = \frac{rc\dot{y}^2}{\overline{cc+yy}^{\frac{3}{2}}}$; wherefore $AT \propto$ $(\frac{\ddot{x}}{\dot{y}^2} =) \frac{rc}{\overline{cc+yy}^{\frac{3}{2}}}$, and $AT = \frac{c^3a}{\overline{cc+yy}^{\frac{3}{2}}}$.

Hence the Curve ST continually approaches nearer and nearer the Hyperbola: And $SQ = (a+x-AT=)$ $a-r+\frac{r}{c}\sqrt{cc+yy} - \frac{c^3a}{\overline{cc+yy}^{\frac{3}{2}}}$, expreſſes the Nature of the Curve ST.

Ex. 5.

Let BA be a Cycloid, $BC = x$, $CA = y$, $BD = s$,
$BV = a$, $\dot{x}$ given. By the Property of the Curve $y =$ 204.

$s+\sqrt{ax-xx}$, but $\dot{s}=\frac{a\dot{x}}{2\sqrt{ax-xx}}$, then $\dot{y}=\frac{2a\dot{x}-2x\dot{x}}{2\sqrt{ax-xx}}=\dot{x}\sqrt{\frac{a-x}{x}}$, and $\ddot{y}=\frac{-a\dot{x}^2}{2x\sqrt{ax-xx}}$; therefore $\frac{-\dot{x}\ddot{y}}{\dot{y}^3}=\frac{r}{2\times\overline{a-x}^2}$: Therefore $AT\propto\frac{1}{\overline{a-x}^2}\propto\frac{1}{\overline{CV}^2}$.

Ex. 6.

203. *Let BA be the Catenary*, $BC=x$, $CA=y$, $BA=z$, $BS=a$, $\dot{x}$ given; then $z=\sqrt{2rx+xx}$, $\dot{z}=\frac{r+x}{\sqrt{2rx+xx}}\dot{x}$, therefore $\dot{y}=(\sqrt{\dot{z}^2-\dot{x}^2}=)\frac{r\dot{x}}{\sqrt{2rx+xx}}$, and $\ddot{y}=\frac{-r\dot{x}^2\times\overline{r+x}}{\overline{2rx+xx}^{\frac{3}{2}}}$; whence $\frac{-\dot{x}\ddot{y}}{\dot{y}^3}=\frac{r+x}{rr}$, therefore AT is as $r+x$, or $AT=\frac{a}{r}\times\overline{r+x}$. Hence,

1. If $a=r$, then $AT=a+x=CS$, and then ST is a right Line paſſing through S.

2. If BS is very ſmall, draw At perpendicular to the Curve, and by ſimilar Triangles $(\dot{z}=)\frac{\overline{r+x}\times\dot{x}}{\sqrt{2rx+xx}}:(\dot{y}=)\frac{r\dot{x}}{\sqrt{2rx+xx}}::(AT=)\frac{a}{r}\times\overline{r+x}:At=a=BS$; therefore the Arch is of the ſame Thickneſs every where: Conſequently a heavy flexible Line put into this Figure would ſupport itſelf.

3. For the Nature of the Curve ST, we have $SQ=(a+x-AT=)\,x-\frac{ax}{r}$. Therefore if a is leſſer than r the Curve is concave towards B, and if a is greater than r, it is convex towards B.

Ex. 7.

Ex. 7.

Let AB be the logarithmic Curve, GD its Aſſymptote, $BS = a$, $BD = r$, Subtangent $GE = t$, $BC = x$, $CA = y$, $AG = r + x$, then by the Property of the Curve $\dot{y} = \frac{t\dot{x}}{r+x}$, and if $\dot{x}$ be given, $\ddot{y} = \frac{-t\dot{x}^2}{\overline{r+x}^2}$, whence $\frac{-\dot{x}\ddot{y}}{\dot{y}^3} = \frac{t\dot{x}^3 \times \overline{r+x}^3}{\overline{r+x}^2 \times t^3\dot{x}^3} = \frac{r+x}{tt}$, whence $AT \propto \overline{r+x}$, or $AT = \frac{a}{r} \times \overline{r+x}$, the ſame as in the laſt Example. FIG. 205.

For the Nature of the Curve, $SQ = (\, a + x - AT =)\, x - \frac{ax}{r}$, therefore if $a = r$, then ST becomes the Aſſymptote DG, and if a be leſs then r, the Curve ST is concave; but if a is greater than r, it is convex towards B.

Or thus for the Nature of the Curve, $GT = (r + x - AT =)\, r - a + \frac{r-a}{r}x$, for the Relation of ST to the Aſſymptote GD.

Ex. 8.

Let AB be the Ciſſoid, whoſe Equation is $axx - yxx = y^3$, in Fluxions $2ax\dot{x} - 2yx\dot{x} - x^2\dot{y} = 3y^2\dot{y}$; this again in Fluxions (making $\ddot{x} = 0$), $2a\dot{x}^2 - 2y\dot{x}^2 - 2x\dot{x}\dot{y} - 2x\dot{x}\dot{y} - x^2\ddot{y} = 6y\dot{y}^2 + 3y^2\ddot{y}$; from the former Equation $\dot{y} = \frac{2ax - 2yx}{3yy + xx}\dot{x}$, and from the latter, $\ddot{y} = \frac{\overline{2a - 2y} \times \dot{x}^2 - 4x\dot{x}\dot{y} - 6y\dot{y}^2}{3yy + xx}$. Now if $a = 10$, $\dot{x} = 1$; ſuppoſe $y = 2$, $x = 1$, to find AT; here $\dot{y} = \frac{16}{13}$, $\ddot{y} = -,5462$; whence $\frac{-\dot{x}\ddot{y}}{\dot{y}^3} = ,293$ for the 206.

Value

FIG. Value of *AT*. And to find it in the Vertex where x and y are 0, and y infinitely greater than x; we ſhall have $axx = y^3$, $\dot{y} = \frac{2ax}{3yy} = \frac{2}{3}\sqrt{\frac{a}{y}}$, $\ddot{y} = \frac{2a - 4x\dot{y} - 6y\dot{y}^2}{3yy} = \frac{-2a}{9yy}$; whence $\frac{-\dot{x}\ddot{y}}{\dot{y}^3} = \frac{3}{4\sqrt{ay}} =$ Infinity.

More generally thus:

Since $axx - yxx = y^3$, in Fluxions $2ax\dot{x} - 2yx\dot{x} - x^2\dot{y} = 3y^2\dot{y}$, whence $\dot{x} = \frac{3xy^2 + x^3}{2y^3}\dot{y}$, this again in Fluxions and reduced (making $\dot{y}$ invariable), $\ddot{x} = \frac{3}{2} \times \frac{\overline{y^3 + x^2y} \times \dot{x} - \overline{xy^2 - x^3} \times \dot{y}}{y^4} \times \dot{y} =$ (expunging xx) $\frac{3}{2}\dot{y} \times \frac{ay\dot{x} - ax\dot{y}}{yy \times \overline{a - y}} =$ (expunging $\dot{x}$) $\frac{3aax\dot{y}^2}{4yy \times \overline{a-y}^2} =$ (expunging x) $\frac{3aa\dot{y}^2}{\overline{a-y}^2 \times 4\sqrt{ay - yy}}$; whence $\frac{\ddot{x}}{\dot{y}^2} = \frac{3aa}{\overline{a-y}^2 \times 4\sqrt{ay-yy}}$: Therefore $AT \propto \frac{1}{\sqrt{y} \times \overline{a-y}^{2\frac{1}{2}}}$.

PROB. XXI.

207. *The Curve BA being given, by whoſe Revolution about the Axis BC there is generated a concave Surface or Vault; To find the Height AT of a Wall ſtanding on the ſame, and ſupported by that Surface, ſo that all the Parts may remain in Equilibrio.*

Let *PBQ* be an infinitely ſmall Part of the Surface contained between the Planes *PBR* and *QBR*, draw the Ordinates *DC*, *AC*; and take *Dd*, *Aa* infinitely ſmall equal Parts of the Curve. Let $BC=x$, $CA=y$, $BA=z$: By the Reaſoning in the laſt Problem, the Weight inſiſting on the ſmall Part of the Surface *ADda* will be as $\frac{1}{R\dot{y}^2}$, when the Particle of the Curve is given. But this incumbent Weight is $ab \times AD \times AT$, But $AD \propto y$, therefore the Weight is as $AT \times y\dot{y}$ and this $\propto \frac{1}{R\dot{y}^2}$, whence $AT \propto \frac{1}{Ry\dot{y}^3}$. Or in general $AT \propto \frac{\dot{z}^3}{Ry\dot{y}^3}$. Whence

Putting $BC=x$, $CA=y$, $BA=z$. *Take* $AT \propto \frac{-\dot{x}\ddot{y}}{y\dot{y}^3}$ *if* $\dot{x}$ *be given, or* $AT \propto \frac{\ddot{x}}{y\dot{y}^2}$ *if* $\dot{y}$ *is given: And the Nature of the Curve ST will be known by expunging x.*

EXAMPLE I.

Let BA be a cubic Parabola, $r^2x=y^3$; then $r^2\dot{x} = 3y^2\dot{y}$, and (if $\dot{y}$ be given) $r^2\ddot{x} = 6y\dot{y}^2$, whence $\frac{\ddot{x}}{y\ddot{y}} = \frac{6}{rr}$; therefore *AT* is as $\frac{6}{rr}$ a given Quantity: Conſequently the Curve *ST* is the ſame Parabola with *BA*, but placed in a higher Poſition. 203.

Ex. 2.

Let BA be a biquadratic Parabola, $r^3x=y^4$, and $\dot{y}$ given. Then $r^3\dot{x} = 4y^3\dot{y}$, and $r^3\ddot{x} = 12y^2\dot{y}^2$, wherefore $\frac{\ddot{x}}{y\dot{y}^2} = \frac{12y}{r^3}$, whence $AT \propto y$ or *AC*. 208.

Ex. 3.

Let BA be a Circle, $y=\sqrt{2rx-xx}$, $\dot{x}$ given; then $\dot{y} = \frac{r\dot{x}-x\dot{x}}{\sqrt{2rx-xx}}$, $\ddot{y} = \frac{-rr\dot{x}^2}{\overline{2rx-xx}|^{\frac{3}{2}}}$; whence 209.

$-\dot{x}\ddot{y}$

FIG. $\frac{-\dot{x}\ddot{y}}{y\dot{y}^3} = \frac{rr}{y\times\overline{r-x}^3}$; therefore $AT \propto \frac{1}{y\times\overline{r-x}^3}$ that is as $\frac{1}{AC\times CR^3}$. Whence the Perpendiculars RS, DE are Aſſymptotes to the Curve ST.

PROB. XXII.

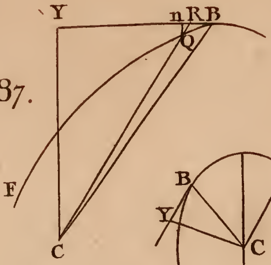
To find the Reſiſtance of a plane Figure or Solid moving in a Fluid, in the Direction of its Axis.

210. Let ABQ be any plane Figure or Solid whoſe Axis is AQ; draw CBf parallel to the Axis AQ, and gD and Ordinate BE Perpendiculars thereto; BD a Tangent at B, and Df perpendicular to it. Call AE, x; EB, y; AB, z; Bn, $\acute{z}$; Br, $\acute{x}$; rn, $\acute{y}$.

Let fB repreſent the Force or Reſiſtance of a Particle of the Fluid, ſtriking againſt C with a given Velocity, then will fD be the Force againſt the Curve Line or Surface at B in Direction fD; and fg will be the Force or Reſiſtance againſt the Curve in Direction BC, which alone is the Reſiſtance that hinders or oppoſes its progreſſive Motion in Direction of the Axis. But by ſimilar Triangles $fB : fg :: fB^2 : fD^2 :: DB^2 : Dg^2 :: \acute{z}^2 : \acute{y}^2$, and $fg = \frac{\acute{y}^2}{\acute{z}^2}\times fB$. Therefore the Force of a Particle againſt C and B are as fB and $\frac{\acute{y}^2}{\acute{z}^2}\times fB$, that is as 1 to $\frac{\acute{y}^2}{\acute{z}^2}$. Now the

Quantity

187.



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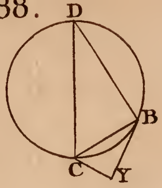

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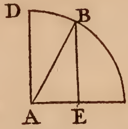

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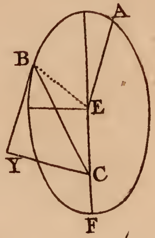

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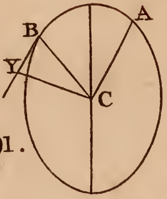

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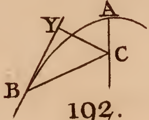

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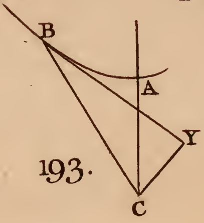

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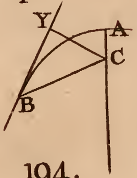

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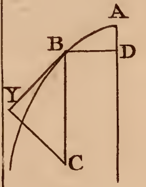

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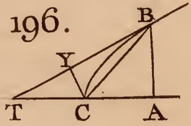

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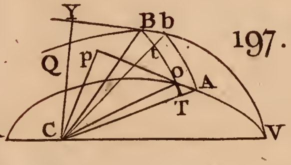

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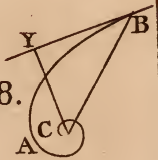

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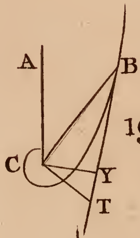

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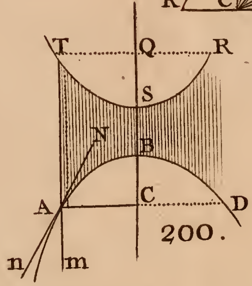

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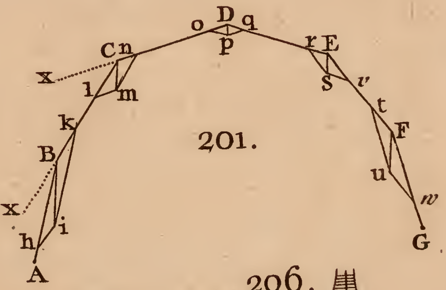

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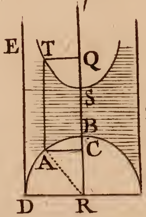

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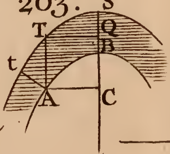

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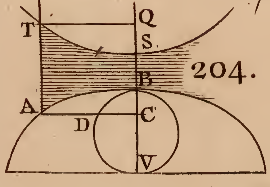

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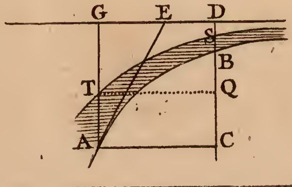

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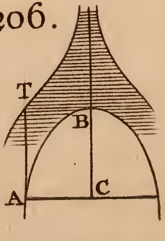

Quantity of Fluid ſtriking againſt *Bn* in the Curve is as $\acute{y}$, and againſt *Bn* in the Solid (generated by *AB* revolving round its Axis) as $y\acute{y}$: Therefore the Force againſt the Baſe : Force againſt the Curve : : $\acute{y}$ to $\frac{\acute{y}^3}{\acute{z}^2}$, or as $\acute{y}$ to $\frac{\acute{y}^3}{\acute{x}^2+\acute{y}^2}$, or as $\acute{y}$ to $\frac{\acute{y}}{1+\frac{\acute{x}^2}{\acute{y}^2}}$, that is as $\dot{y}$ to $\frac{\dot{y}}{1+\frac{\dot{x}^2}{\dot{y}^2}}$: And Force againſt the Baſe : Force againſt the Solid, is as $y\dot{y}$ to $\frac{y\dot{y}}{1+\frac{\dot{x}^2}{\dot{y}^2}}$. FIG. 210.

Note, by the Reſiſtance of a plane Figure moving in a Fluid is meant the Reſiſtance of a priſmatic Solid of any given Depth, and whoſe Baſe is that Figure: And it is ſuppoſed to move in a Direction parallel to that Baſe.

Hence to find the Reſiſtance; by the Equation of the Curve, exterminate $\dot{x}^2$ *out of the Quantity* $\frac{\dot{y}}{1+\frac{\dot{x}^2}{\dot{y}^2}}$ *for the Curve; and find the Fluent F; or out of the Quantity* $\frac{y\dot{y}}{1+\frac{\dot{x}^2}{\dot{y}^2}}$ *for the Solid, and find its Fluent G. Then will the Reſiſtance againſt the Baſe : to the Reſiſtance againſt the Curve : : y : F. Or the Reſiſtance againſt the Baſe : to the Reſiſtance againſt the Solid : : as* y^2 *: to* 2*G.*

EXAMPLE I.

Let there be a Triangle (or priſmatic Solid) ARS moving in Direction QA. Let AQ=b, QR=c; AR=d, 211.

FIG.

$AE = x$, $EB = y$; and by ſimilar Triangles $x = \frac{by}{c}$, whence $\dot{x}^2 = \frac{bb\dot{y}^2}{cc}$, therefore $\frac{\dot{y}}{1+\frac{\dot{x}^2}{\dot{y}^2}} = \frac{\dot{y}}{1+\frac{bb}{cc}} = \frac{cc\dot{y}}{cc+bb} = \frac{cc\dot{y}}{dd}$, whoſe Fluent $F = \frac{ccy}{dd}$: Whence the Reſiſtance of the Baſe to that of the Side, is as y to $\frac{ccy}{dd}$, that is as dd to cc.

Ex. 2.

211. *Let ARS be a Cone*, whoſe Axis is AQ; then $x = \frac{by}{c}$, $\dot{x}^2 = \frac{bb\dot{y}^2}{cc}$, and $\frac{y\dot{y}}{1+\frac{\dot{x}^2}{\dot{y}^2}} = \frac{y\dot{y}}{1+\frac{bb}{cc}} = \frac{ccy\dot{y}}{cc+bb} = \frac{ccy\dot{y}}{dd}$, whoſe Fluent $G = \frac{ccyy}{2dd}$, and $2G = \frac{ccyy}{dd}$. Therefore the Reſiſtance of the Baſe to that of the Side is as yy to $\frac{ccyy}{dd}$, or as dd to cc.

Ex. 3.

210. *Let ABR be a Circle*, Radius $AQ = r$, $AE = x$, $EB = y$. Then $\dot{x} = \frac{y\dot{y}}{r-x} = \frac{y\dot{y}}{\sqrt{rr-yy}}$, and $\frac{\dot{x}^2}{\dot{y}^2} = \frac{yy}{rr-yy}$; therefore $\frac{\dot{y}}{1+\frac{\dot{x}^2}{\dot{y}^2}} = \frac{\dot{y}}{1+\frac{yy}{rr-yy}} = \frac{rr-yy}{rr}\dot{y}$, whoſe Fluent $F = y - \frac{y^3}{3rr}$. Whence the Reſiſtance againſt the Baſe, to the Reſiſtance againſt the Circle (or cylindric Surface) is as y to $y - \frac{y^3}{3rr}$, that is as $3rr$ to $3rr - yy$: which when $y = r$, is as 3 to 2.

Ex. 4

Ex. 4.

Let ABQ be an Hemiſphere, then $\frac{y\dot{y}}{1+\frac{\dot{x}^2}{\dot{y}^2}} =$ FIG. 210. $\frac{rr-yy}{rr}\times y\dot{y}$, whoſe Fluent $G = \frac{y^2}{2} - \frac{y^4}{4rr}$; therefore the Reſiſtance againſt the Baſe, to the Reſiſtance againſt the convex Surface, is as y^2 to $y^2 - \frac{y^4}{2rr}$, or as $2rr$ to $2rr-yy$; which when $y=r$, is as 2 to 1.

Ex. 5.

Let $ABRQ$ be a Spheroid, Q the Center, $QA=a$, 210.
Latus Rectum $=2r$, $AE=x$, $BE=y$, $QE=u=a-x$; then $uu=aa-\frac{a}{r}yy$, and $\dot{x}=-\dot{u}=\frac{ay\dot{y}}{r\sqrt{aa-\frac{a}{r}yy}}$; therefore $\frac{y\dot{y}}{1+\frac{\dot{x}^2}{\dot{y}^2}} = \frac{rra-ryy}{rra+\overline{a-r}\times yy}y\dot{y}$; whoſe Fluent, by Form the 4th and 11th, is $G=\frac{-ryy}{2a-2r} + \frac{\frac{1}{2}rraa\times 2.30258}{\overline{a-r}^2}\times \text{Log}:\frac{rra+\overline{a-r}\times yy}{rra}$: Whence the Reſiſtance of the Baſe to the Reſiſtance of the convex Surface, is as yy to $2G$; and when $y=QR$, it will be as $a-r$ to $\frac{ar}{a-r}\times\text{Log}:\frac{a}{r}:-r$.

Ex. 6.

Let ABE be an Hyperboloid; denoting the Quan- 210.
tities as in the laſt Example, $uu=aa+\frac{a}{r}yy$, and

FIG. 210. $\frac{y\dot{y}}{1+\frac{\dot{x}^2}{\dot{y}^2}} = \frac{rra+ryy}{rra+\overline{r+a}\times yy}\,y\dot{y}$, whoſe Fluent $G =$

$\frac{1}{2}\times\frac{ryy}{r+a}+\frac{1}{2}\times\frac{rraa}{\overline{r+a}^{2}}\times 2.30258$ Log. $\frac{rra+\overline{a+r}\times yy}{rra}$.

Therefore Reſiſtance of the Baſe : Reſiſtance of the convex Surface : : yy to $2G$.

Ex. 7.

210. *Let ABE be a Paraboloid,* $AE = x$, $BE = y$, $2rx = yy$, then $\dot{x}^2 = \frac{y^2\dot{y}^2}{rr}$; and $\frac{y\dot{y}}{1+\frac{\dot{x}^2}{\dot{y}^2}} =$ $\frac{rry\dot{y}}{rr+yy}$, whoſe Fluent $G = 2.3025rr$ Log : $\sqrt{rr+yy}$:

And by Correction $G = 2.302585rr \times$ Log : $\frac{\sqrt{rr+yy}}{r}$, And the Reſiſtance of the Baſe, to the Reſiſtance of the Surface of the Solid, is as yy to $2.30258rr \times$ Log : $\frac{rr+yy}{rr}$:

Ex. 8.

210. *Suppoſe AB to be a cubic Parabola,* $r^2x = y^3$, then $\dot{x}^2 = \frac{9y^4\dot{y}^2}{r^4}$, whence $\frac{y\dot{y}}{1+\frac{\dot{x}^2}{\dot{y}^2}} = \frac{r^4y\dot{y}}{r^4+9y^4}$, whoſe Fluent G (by Form the 5th) $= \frac{,01745rr}{6}$ $\times$ Degrees in the Arch whoſe Tangent is $\frac{3yy}{rr}$; and the Reſiſtance of the Baſe : Reſiſtance of the Surface : : as yy to $2G$.

Ex. 9.

212. *Let ABDQ be the Solid generated by the Cycloid ABD revolving round AQ.* $DQ = a$, $AE = x$, $EB = y$

$EB = y$, $AB = z$; by the Nature of the Curve, $y = z - \frac{zz}{4a}$, and $\dot{y} = \dot{z} - \frac{z\dot{z}}{2a}$; then $\frac{y\dot{y}}{1 + \frac{\dot{x}^2}{\dot{y}^2}}$ FIG. 212.
$= \frac{y\dot{y} \times \dot{y}^2}{\dot{z}^2} = \frac{y\dot{y} \times \overline{2a - z}^2}{4aa} = \frac{\overline{a - y} \times y\dot{y}}{a}$, whoſe Fluent $G = \frac{yy}{2} - \frac{y^3}{3a}$: Whence the Reſiſtance of the Baſe, to the Reſiſtance of the Surface : : as yy, to $yy - \frac{2y^3}{3a}$, or as 1 to $1 - \frac{2y}{3a}$, and in the whole Solid it is as 3 to 1.

PROB. XXIII.

To find the Center of Gyration of any Body.

This Center is ſuch a Point *O*, that a given Force acting at a given Diſtance from the Axis of Rotation *SR*, will in the ſame Time generate the ſame angular Velocity in that Body, as it would do if the whole Body was placed in *O*. 239.

Let *A*, *B*, *C*, &c. be any Particles of the Body, and let fall the Perpendiculars *Aa*, *Bb*, *Cc*, &c. and *Or*, *Pp*, on the Axis of Motion *SR*. And ſuppoſe a given Force *S* apply'd at *P*, acting at the given Diſtance *Pp* from that Axis, to move the Body about *SR*. The abſolute Motion of the Particle *A* being as $Aa \times A$; the Force acting at *A* that generates it muſt alſo be as $Aa \times A$; and a Force acting at *P* that will generate the ſame Motion in *A*, is as $\frac{Aa}{Pp} \times Aa \times A$, or $\frac{Aa^2 \times A}{Pp}$. Likewiſe the Forces at *P* that gene-

FIG. 239. rates the Motions of *B* and *C*, will be as $\frac{Bb^2\times B}{Pp}$ and $\frac{Cc\times C}{Pp}$, *&c.* and the Sum of all theſe $= S$. And the Force at *P* that can generate the ſame angular Motion of *A*, placed in *A*, will be as $\frac{Or^2\times A}{Pp}$, likewiſe the Force at *P* that can generate the Motion of the whole Body, placed in *O*, will be $\frac{Or^2\times \text{Body}}{Pp} = S$, by the Hypotheſis. Therefore the Sum of all the $\frac{Aa^2\times A}{Pp} + \frac{Bb^2\times B}{Pp} + \frac{Cc^2\times C}{Pp}$, *&c.* $= \frac{Or^2\times \text{Body}}{Pp}$, whence $Or^2 = \frac{Aa^2\times A + Bb^2\times B, \&c.}{\text{Body}}$. Therefore if $z = $ Body, $x =$ Diſtance of any Particle, then $rO^2 = \frac{\text{Sum of } xx\dot{z}}{z} = \frac{\text{Fluent of } xx\dot{z}}{\text{Body}}$. Whence this

RULE.

Multiply the Fluxion of the Figure by the Square of the Diſtance of the generating Point, Line, or Surface, from the Axis of Motion, and find the Fluent; which divided by the whole Body, the Square Root of the Quotient, is the Diſtance of the Center of Gyration.

EXAMPLE. I.

240. *Let SA be a right Line moving round S.* Let $SB = x$, $SA = a$, then $\dot{z} = \dot{x}$, and Fl. $x^2\dot{x} = \frac{x^3}{3}$, and $z = x$, then $\sqrt{\frac{x^3}{3x}}$ or $x\sqrt{\tfrac{1}{3}}$ or $a\sqrt{\tfrac{1}{3}} =$ Diſtance of the Center of Gyration *SO*; the ſame is true of a ſlender Cylinder.

Ex. 2.

Ex. 2.

In the Periphery of a Circle revolving about the Diameter SR. Let $CA = r$, $BA = x$, $SA = z$, $c =$ Circumference, then $\dot{z} = \frac{r\dot{x}}{\sqrt{rr-xx}}$. And the whole Fluent of $\frac{rx^2\dot{x}}{\sqrt{rr-xx}}$ (by Form 17) $= \frac{rrz}{2} = \frac{1}{4}c \times \frac{rr}{2}$; and for the whole Circumference it is $\frac{crr}{2}$, then $\sqrt{\frac{crr}{2c}} = r\sqrt{\frac{1}{2}} = CO$, the Diſtance of the Center of Gyration. FIG. 241.

Ex. 3.

For the Plane of a Circle, or a Cylinder, revolving round the Axis in C. Let $CA = r$, $CB = x$, $c = 3.1416$, then Circumference at $B = 2cx$, and $\dot{z} = 2cx\dot{x}$, and Fluent $2cx^3\dot{x} = \frac{1}{2}cx^4$; and for the whole Circle, 'tis $\frac{1}{2}cr^4$; but the Area of the Circle $= crr$, whence $\sqrt{\frac{cr^4}{2crr}} = r\sqrt{\frac{1}{2}}$, the Diſtance CO. 242.

Ex. 4.

For the Plane of the Circle about the Diameter RS. Let $DC = r$, $CB = x$, then Flux. Figure $= \dot{x}\sqrt{rr-xx}$, and the whole Fl. of $x^2\dot{x}\sqrt{rr-xx}$, when $x = r$, is $\frac{1}{4}rr \times$ Quadrant; and for the whole Circle, $\frac{1}{4}rr \times$ Circle, this divided by the Circle gives $\frac{1}{4}rr$, and $\sqrt{\frac{1}{4}rr}$ or $\frac{1}{2}r =$ Diſtance CO. 243.

Ex. 5.

For the Surface of a Sphere about the Diameter RS. Let $DC = r$, $CB = x$, $RD = s$, $c = 3.1416$. Then the Circumference of the Circle deſcribed by D is $= 2cx$, and $2cx\dot{s} =$ Fluxion of the Figure $= \frac{2cx \times r\dot{x}}{\sqrt{rr-xx}}$, 243.

multiply

FIG.

multiply by xx and we have $\frac{2crx^3\dot{x}}{\sqrt{rr-xx}}$; but when $x=r$, whole Fluent of $\frac{x\dot{x}}{\sqrt{rr-xx}}=r$, and (by Form 17) the whole Fluent of $\frac{2crx^3\dot{x}}{\sqrt{rr-xx}}=\frac{4}{3}cr^4$, and for the whole Surface tis $=\frac{8}{3}cr^4$, and the Surface of the Sphere $=4crr$, whence $\frac{8cr^4}{3\times 4crr}=\frac{2rr}{3}$, and $\sqrt{\frac{2}{3}rr}$ or $r\sqrt{\frac{2}{3}}=CO$ the Diſtance required.

Ex. 6.

243. *Let RAS be a Globe revolving about the Diameter* RS; let $CD=r$, $CB=x$, $c=3.1416$; then $2cx\times 2DB\times\dot{x}$ or $4cx\dot{x}\sqrt{rr-xx}=\dot{z}=$ Fluxion of the Solid; and the Fluent of $4cx^3\dot{x}\sqrt{rr-xx}=\frac{8}{15}crr\times r^3$ (by Form 17) $=\frac{8}{15}cr^5$, when $x=r$. And the Globe $=\frac{4}{3}cr^3$; whence $\sqrt{\frac{2}{5}rr}$ or $r\sqrt{\frac{2}{5}}=$ Diſtance CO.

Ex. 7.

244. *Let ACB be a Cone revolving about the Axis AB.* Let $AB=a$, $BC=b$, $BE=x$, $c=3.1416$; then $DE=\frac{ab-ax}{b}$. And $\dot{z}=2cx\dot{x}\times\frac{ab-ax}{b}$, and multiplying by xx, we have $2cax^3\dot{x}-\frac{2cax^4\dot{x}}{b}$, whoſe Fluent is $\frac{cax^4}{2}-\frac{2cax^5}{5b}=\frac{1}{10}cab^4$, when $x=b$. The Solidity of the Cone is $=\frac{1}{3}cabb$, then $\sqrt{\frac{3cab^4}{10cabb}}=b\sqrt{\frac{3}{10}}=BO$.

Ex. 8.

245. *Let AD be a Paraboloid revolving about the Axis* AB. Put $AB=a$, $BC=b$, $AF=s$, $FD=x$, $c=3.1416$, and $rs=xx$. Then $DE=a-\frac{xx}{r}$; and $\dot{z}=2cx\dot{x}$

×

$\times a - \frac{xx}{r}$, which multiply'd by xx is $2cax^3\dot{x} -$ FIG. 245.
$\frac{2cx\dot{x}}{r}$, whoſe Fluent is $\frac{cax^4}{2} - \frac{cx^6}{3r} = \frac{1}{6}cab^4$ when $x = a$: And Paraboloid $= \frac{1}{2}cabb$, therefore $\frac{2cab^4}{6cabb} = \frac{1}{3}bb$, and $\sqrt{\frac{1}{3}bb} = b\sqrt{\frac{1}{3}} =$ Diſtance BO.

PROB. XXIV.

To find the lateral Strength of a Piece of Timber, whoſe Section is any Figure given.

Suppoſe a Beam FA fixt with one End in a Wall, 246.
and a Weight P ſuſpended at the other End F to break it, and let ABD be the Section of it where it breaks. Let the abſolute Strength of 1 Fibre of the Wood be 1, and put $AB = a$, $BD = b$, $AC = x$, $CE = y$, $BC = v$, and $BF = 1$. When the Beam breaks, the Parts at B don't ſeparate at all, and the Parts at A ſeparate the firſt, and the Degree of ſtretching at any Plane C will be as BC. But by Mechanics the Reſiſtance any Particle C makes, will be as the Degree of ſtretching, and therefore as BC. Hence $a : 1 :: v : \frac{v}{a} =$ the Reſiſtance or Tenſion of a Fibre at C; and by reaſon of the bended Leaver ABF, whoſe Fulcrum is in F; $1 : v :: \frac{v}{a} : \frac{vv}{a}$ the effect of that Tenſion upon the Weight P; or the Weight ſuſpended at F to ballance that Reſiſtance; and therefore $\frac{vvy}{a} =$ Weight at F to ballance the Reſiſtance of all the Fibres

FIG. 246. bres in the Line CE; that is $\frac{vyy}{a}$ = Strength of all the Particles in CE. Therefore the Sum of all the $\frac{vvy}{a}$, in the Figure, or the Fluent of $\frac{vvy\dot{x}}{a}$ or $\frac{yv^2\dot{v}}{a}$ will = whole Weight P or the Strength of the Beam.

RULE.

Expunge v *or* y *or* $\dot{x}$ *by the Nature of the Figure, and find the Fluent of* $\frac{yvv\dot{x}}{a}$ *or of* $\frac{yv^2\dot{v}}{a}$; *and that will be as the Strength of the Beam.*

COR. If that Fluent be divided by the Section (or Fl. $y\dot{x}$ or $v\dot{v}$), the Quotient will ſhow the Diſtance from B, where the total Strength of all the Fibres of the Beam being collected, it will be equally ſtrong.

EXAMPLE I.

246. *If ABD be a Parallelogram*; then $y=b$, and $\frac{yv^2\dot{v}}{a} = \frac{bv^2\dot{v}}{a}$; whoſe Fluent is $\frac{bv^3}{3a}$ = (when $v=a$) $\frac{1}{3}baa$, for the Strength, $= \frac{1}{3}a \times$ Section.

EX. 2.

Let ABD be a Triangle; then $y = \frac{bx}{a}$, and $v = a - x$; whence $\frac{yv^2\dot{x}}{a} = \frac{bx\dot{x}}{aa} \times \overline{aa - 2ax + xx}$. And the Fluent is $\frac{bax^2}{2a} - \frac{2bx^3}{3a} + \frac{bx^4}{4aa} = \frac{1}{12}ba^2$, when $x=a$, for the Strength, $= \frac{1}{6}a \times$ Section.

If the Vertex be at B, then $y = \frac{bv}{a}$, and $\frac{yv^2\dot{v}}{a}$ =

FIG. 246.

$= \frac{bv^3\dot{v}}{aa}$, and the Fluent $= \frac{bv^4}{4aa} = \frac{1}{4}baa$, when $v = a$, for the Strength, or $\frac{1}{2}a \times$ Section = Strength.

Ex. 3.

If AEB be a Circle; and $y = \sqrt{ax - xx} = \sqrt{av - vv}$, and $\frac{yv^2\dot{v}}{a} = \frac{v^2\dot{v}}{a}\sqrt{av - vv}$; but $F : \dot{v}\sqrt{av - vv}$ or $v^{\frac{1}{2}}\dot{v}\sqrt{a - v}$ = Semicircle: And by Form 17 the whole Fluent of $v^2\dot{v}\sqrt{av - vv}$, or $v^{2\frac{1}{2}}\dot{v}\sqrt{a - v} = \frac{5}{16}aa \times$ Semicircle, and when doubled, $F : \frac{yv^2\dot{v}}{a} = \frac{5}{16}a \times$ Circle $= \frac{5 \times 3.1416}{16 \times 4}a^3$, for the Strength of the whole circular Section $= \frac{5}{16}a \times$ Section.

Ex. 4.

In a hollow Cylinder, or the Periphery of the Circle AEB. Let the Arch $= z$, then $y\dot{v} = \dot{z}$, and $\dot{z} = \frac{\frac{1}{2}a\dot{v}}{\sqrt{av - vv}} = \frac{1}{2}a \times \frac{v^{-\frac{1}{2}}\dot{v}}{\sqrt{a - v}}$, therefore Fl. of $\frac{v^{-\frac{1}{2}}\dot{v}}{\sqrt{a - v}} = \frac{z}{\frac{1}{2}a}$. And by Form 17, whole Fluent of ($\frac{yv^2\dot{v}}{a}$ or) $\frac{1}{2} \times \frac{v^2\dot{v}}{\sqrt{av - vv}}$ or $\frac{1}{2} \times \frac{v^{1\frac{1}{2}}\dot{v}}{\sqrt{a - v}}$, is $= \frac{3aa}{8a} \times$ Semicircle, when $v = a$; and for the whole Periphery $= \frac{3}{8}a \times$ Circle $= \frac{3 \times 4.1416}{8}a^2$, the Strength; or $\frac{3}{8}a \times$ Section = the Strength.

The Problems delivered in this Section are exceeding general, each of them comprehending an infinite Number of particular Caſes, and are ſufficient

FIG. here to ſhew the Method of inveſtigating general Problems by the Method of Fluxions. I ſhall now proceed to exemplify the ſame Doctrine, in the Reſolution of a few particular Problems, belonging to Phyſicks or Natural Philoſophy; and the rather becauſe this Sort of Problems has not been ſo common among the Writers of Fluxions.

* * *

SECT.

SECT. III.

The Solution of Phyſical Problems, or ſuch as occur in the Phænomena of Nature.

PROB. I.

To find the Relation of the Fluxions, of the Times, Velocities, and Spaces deſcribed by Bodies in Motion; being acted upon by any accelerating Forces.

LET b = *Body or Quantity of Matter.*
m = *Motion generated.*
F = *Accelerative Force.*
t = *Time of moving.*
v = *Velocity.*
s = *Space deſcribed.*

Let $\acute{t}$ = Moment of Time, or an exceeding ſmall Part of Time, $\acute{s}$ = Moment of Space, $\acute{v}$ = Moment of Velocity, $\acute{m}$ = Increment of Motion Then whatever be the Law of the accelerating Force F, it may

FIG. be conſidered as uniform for the Moment of Time $\acute{t}$. Then by the common Mechanics, the Velocity generated in the Time $\acute{t}$ will be as the Force and the Time directly, and Quantity of Matter reciprocally; that is $\acute{v} \propto \frac{F\acute{t}}{b}$. Likewiſe the Velocity may be look'd upon as uniform, for the Moment of Time $\acute{t}$; therefore by the Laws of uniform Motion, the Space deſcribed will be as the Time and Velocity, or $\acute{s} \propto v\acute{t}$. Alſo the Motion generated is as the Force and Time, that is $\acute{m} \propto F\acute{t}$. Now $\acute{s}$, $\acute{t}$, $\acute{v}$, $\acute{m}$, being ſuppoſed infinitely ſmall; ſubſtitute the Fluxions inſtead of the Moments, and we have $\dot{m} \propto F\dot{t}$, $\dot{v} \propto \frac{F\dot{t}}{b}$, and $\dot{s} \propto v\dot{t}$, univerſally. And ſince by Mechanics m is as bv, therefore $\dot{m} \propto b\dot{v}$. Hence from theſe Equations we ſhall have,

1. $\dot{m} \propto F\dot{t} \propto b\dot{v}$, and $\dot{t} \propto \frac{\dot{m}}{F}$.

2. $\dot{s} \propto v\dot{t}$, and $\dot{t} \propto \frac{\dot{s}}{v}$.

3. $\dot{v} \propto \frac{F\dot{t}}{b}$, and $\dot{t} \propto \frac{b\dot{v}}{F}$.

4. $v\dot{v} \propto \frac{F\dot{s}}{b}$, and $\dot{s} \propto \frac{bv\dot{v}}{F}$.

And in any of theſe Forms, if b or F be conſtantly the ſame; then ſuch conſtant Quantity or Quantities muſt be left out. Likewiſe if the Velocity or the Space be decreaſing, you muſt write $-\dot{v}$, or $-\dot{s}$ inſtead of $\dot{v}$, $\dot{s}$.

COR. *Hence, put* $h = 16\frac{1}{12}$ *Feet, the Space deſcended by* b, *in one Second; then* $2h =$ *Velocity acquired in one Second. Then will* $\dot{v} =$

$$\left.\begin{array}{l} \dot{v} = \dot{s} \text{ or } 2b\dot{t} \\ \dot{s} = \dot{v} \text{ or } 2b\dot{t} \\ \dot{t} = \dfrac{\dot{s}}{2b} \text{ or } \dfrac{\dot{v}}{2b} \end{array}\right\} \textit{when } v=2b, \textit{ and } F=b\textit{; which is the Caſe of falling Bodies.}$$

For the Velocity generated is as the Time, therefore $\overset{\prime\prime}{1}$ (Time) : $2b$ (Velocity) : : $\overset{\prime}{t} : \overset{\prime}{v} : : \dot{t} : \dot{v}$, and $\dot{v} = 2b\dot{t}$.

Likewiſe the Space deſcribed with the uniform Velocity $2b$, is as the Time; therefore, $2b$ (Space) : $\overset{\prime\prime}{1}$ (Time) : : $\overset{\prime}{s} : \overset{\prime}{t} : : \dot{s} : \dot{t}$, and $\dot{s} = 2b\dot{t}$.

Theſe Forms are general, and will be found ſerviceable in the Solution of many phyſical and mechanical Problems.

PROB. II.

To find the Motion of a muſical String, vibrating at very ſmall Diſtances.

1. Let *AB* be the String, and let it be drawn to *C*, and there let go; now ſince the Force to move the Point *C*, by Mechanics, is as the Sine of the Angle *ACn*, or as that Angle itſelf when it is very ſmall; therefore the Point *C* alone will firſt begin to move, and preſently by the Flexure of the String in *d* and *e* theſe Points will alſo begin to move, and then the next Points to theſe, and ſo forward. Now by reaſon of the great Flexure in *C*, that Point will at firſt be very ſwiftly moved; and the Curvature in *d* and *e* being thereby increaſed, theſe Points are continually accelerated; and the Curvature in *C* being diminiſhed, its 214.

FIG. 214. its Motion will be leſs accelerated. And univerſally theſe Points that are too ſlow being more accelerated, and thoſe too ſwift being leſs accelerated, it will come to paſs that, the Forces being at length rightly adjuſted, all the Points of the String will acquire ſuch Motions, as to be carried to the Axis together; and will continue to go and return together *ad infinitum.*

2. Now that this may be regularly performed, the String muſt always have the Form of the Curve *AFXB*, whoſe Nature is ſuch, that the Angle of Contact, or the Curvature in any Point *F*, will be as the Ordinate *FE*; for then the Force at *F* being as the Curvature that is as *FE*, the Velocity generated will alſo be as *FE* the Space to be deſcribed; and the conſtant Accelerations and Velocities, and the Parts of the Ordinate deſcribed, and thoſe to be deſcribed, will be as the wholes; and conſequently any correſpondent Parts of the Ordinates, and therefore the whole Ordinates, will be deſcribed in equal Times.

3. To find the Radius of Curvature; Let AB or $2AZ = a$, $ZX = b$; $AE = x$, $EF = y$, $AF = z$, $e =$ Radius of Curvature in the middle Point X. Let $\dot{z}$ be given; and by Prob V. Sect. II. the Radius of Curvature in F $= \frac{\dot{z}\dot{y}}{\ddot{x}}$; therefore by the Nature of the Curve $y : b :: e : \frac{be}{y} = \frac{\dot{z}\dot{y}}{\ddot{x}}$, and $eb\ddot{x} = \dot{z}y\dot{y}$, and the Fluent is $eb\dot{x} = \frac{yy\dot{z}}{2}$, but in X, $\dot{x} = \dot{z}$, and $y = b$: Therefore the Fluent corrected is $eb\dot{x} - eb\dot{z} = \frac{yy - bb}{2}\dot{z}$, and $eb\dot{x} = eb\dot{z} + \frac{y - bb}{2}\dot{z} = \frac{2eb + yy - bb}{2}\sqrt{\dot{x}^2 + \dot{y}^2}$, and by Reduction, $\dot{x} = \frac{\overline{2eb + yy - bb} \times \dot{y}}{\sqrt{4eb^3 - 4cby^2 - y^4 - b^4 + 2bbyy}}$

=

FIG. 214.

$= $ (becauſe e is vaſtly greater than b or y) $\frac{2eb\dot{y}}{\sqrt{4eb^3-4ebyy}}$ $= \frac{\dot{y}\sqrt{eb}}{\sqrt{bb-yy}}$. Whence (by Form the 10th) the Fluent $x = \sqrt{eb} \times$ Arch whoſe Sine is $\frac{y}{b}$, Radius $= 1$; and when $y = b$, then $x = \frac{1}{2}a$, and then $x = \sqrt{eb} \times \frac{3.1416}{2} = \frac{c}{2}\sqrt{eb}$, putting $c = 3.1416$. Therefore $\frac{1}{2}a = \frac{c}{2}\sqrt{eb}$, whence $e = \frac{aa}{ccb}$, the Radius of Curvature in X. Therefore the Radius of Curvature in $F = \frac{aa}{ccy}$.

4. To find the Motion of any Particle of the String as ſuppoſe of X the middle Point. Let $p =$ Tenſion of the String, or the Force that extends it; $n =$ Weight of the String; $IZ = x$, $v =$ Velocity in I, $t =$ Time of deſcribing XI; $\acute{x}$, $\acute{v}$, $\acute{t}$, the Moments of x, v, and t. The Radius of Curvature in I is $= \frac{aa}{ccx}$. By Example 16, Prop. XIII. The Force wherewith any Particle of the Curve at I is urged; is to the Tenſion of the String (p) : : as that Particle ($\acute{z}$) : to the Radius of Curvature in I ($\frac{aa}{ccx}$): therefore the Force acting at $I = \frac{pccx\acute{z}}{aa}$. And hence the Force upon the Particle $\acute{z}$ is as the Diſtance x: Now by Mechanics or the Laws of Motion, the Velocity $\times$ Moment of Velocity $\propto \frac{\text{Moment of Space} \times \text{Force}}{\text{Weight}}$ (ſince the Weight is as the Matter); which is a univerſal

FIG. 214. verſal Proportion for theſe Quantities. (See Prob. I.)

Now it is known that any heavy Body falling thro' $16\frac{1}{12}$ or f Feet gains a Velocity of $2f$ in 1 Second; therefore $\acute{x}$ = Velocity generated by that Body in falling thro' $\acute{x}$ with that Velocity $2f$, becauſe the Velocities generated are as the Times, or as the Spaces ($2f$ and $\acute{x}$) uniformly deſcribed with the given Velocity $2f$. Therefore, in this Caſe of falling Bodies, we have the Velocity × Moment of Velocity $= 2f\acute{x}$: and likewiſe $\frac{\text{Force} \times \text{Moment of Space}}{\text{Weight}} = \frac{p\acute{x}}{p} = \acute{x}$. Laſtly in the Caſe of the vibrating String $v\acute{v}$ = Velocity × Moment of Velocity. And, becauſe the String is homogeneous $a : n :: \acute{z} : \frac{n\acute{z}}{a}$ = Weight of $\acute{z}$; and $-\acute{x}$ = Moment of Space. Therefore (by the Rules, Prop. XIII.) we get this Analogy (from the general Proportion before laid down) $2f\acute{x} : \acute{x} :: v\acute{v} : \frac{-\acute{x} \times pccx\acute{z}}{aa \times \frac{n\acute{z}}{a}}$. Whence $v\acute{v} = \frac{-2fpccx\acute{x}}{na}$, or $vv = \frac{-2fpccx\dot{x}}{na}$; whence the Fluent is $vv = \frac{-2fpccx^2}{na}$: But in X, $v=0$, $x=b$; therefore the Fluent corrected (by Prop. XII.) is $vv = \frac{2fpcc}{na} \times \overline{bb - xx}$. And in Z, where $x = 0$, $v = bc\sqrt{\frac{2fp}{na}}$, the Feet deſcribed in a Second.

5. Laſtly for the Time. Since the Moment of the Time $\propto \frac{\text{Moment of Space}}{\text{Velocity}}$, univerſally. And in a alling Body, $2f$ (Space) : 1 Second (Time) : : $\dot{x}$ (Space) : $\frac{\dot{x}}{2f} \doteq$ Time of deſcribing $\dot{x}$ by the falling Body. And likewiſe $\frac{\dot{x}}{2f} = \frac{\text{Space}}{\text{Velocity}}$; therefore from he general Analogy, we get $\dot{t} = \frac{-\dot{x}}{v}$, or $\dot{t} =$

$$\frac{-\dot{x}}{c\sqrt{\frac{2fp}{na}} \times \sqrt{bb-xx}}, \text{ then the Fluent } t = \frac{-A}{c\sqrt{\frac{2fp}{na}}}$$

putting $A =$ Arch whoſe Sine is $\frac{x}{b}$ and Radius 1). And by Correction (for in X, $t = 0$, $x = b$;) $t =$ $\frac{B}{c\sqrt{\frac{2fp}{na}}}$ (putting $B =$ Arch whoſe Coſine is $\frac{x}{b}$); nd when $x = 0$, the whole Time $t = \frac{1}{2}\sqrt{\frac{na}{2pf}}$; and t or the Time of one entire Vibration $= \sqrt{\frac{na}{2pf}}$, in econds.

FIG. 214.

Cor. 1. Hence all the Vibrations great and ſmall e performed in equal Times; for they are all expreſſed y $\sqrt{\frac{na}{2pf}}$, in which b is not concerned.

Cor. 2. The Number of Vibrations performed in ne Second is $\sqrt{\frac{2fp}{na}}$.

Cor. 3. Hence the Square of the Time of Vibra- on of any muſical String, is as it's Length and eight directly, and it's Tenſion reciprocally. And

FIG. 214. therefore in the ſame String and Tenſion the Time is as the Length.

SCHOLIUM.

To conſtruct the *harmonical Curve AFX*; let s be the Arch of a Circle, whoſe Radius is b, or XZ; d the Diameter of another Circle, whoſe Circumference is a or AB. Make AE or $x = \frac{ds}{b}$, and erect $EF = y$, the Sine of s; and F is in the Curve.

For we found $\dot{x} = \frac{\dot{y}\sqrt{eb}}{\sqrt{bb - yy}}$, and $e = \frac{aa}{ccb}$, therefore $\dot{x} = \frac{a\dot{y}}{c\sqrt{bb - yy}} = \frac{d}{b} \times \frac{b\dot{y}}{\sqrt{bb - yy}} =$ (by the Nature of the Circle) $\frac{d\dot{s}}{b}$; whence $x = \frac{ds}{b}$: Therefore AFB is a mechanical Curve.

I took a virginal String $29\frac{56}{100}$ Inches long, and weighing $8\frac{6}{10}$ Grains; and faſtning it to the Virginal, I ſtretched it with $8\frac{1}{2}$ Pound Weight Avoirdupoiſe; and cauſing it to vibrate, I found it to be Uniſon with the Note Ela in the Baſe (the Note below the Cliff): By this Problem it appears, that the String made 300 Vibrations in a Second of Time. This Experiment I made very accurately. However, by Reaſon of the Reſiſtance of the Air, and the larger Vibrations that the String makes, it is probable that the Time is a little prolonged; and that the Number of Vibrations in a Second may be ſomething leſs than is aſſigned by this Problem.

PROB

PROB. III.

To find the Velocity of a Projectile at A moving in any given Curve QAO about the Center of Force S. FIG. 215.

Let the Diſtance $SA = D$, SB the Perpendicular on the Tangent at $A = P$. Radius of Curvature $CA = R$, $c =$ Velocity of the Body at A, $e =$ Velocity of a Body in a Circle at the ſame Diſtance SA, and acted on with the ſame Force; take the infinitely ſmall Arch Aa, and draw am, an, parallel to SA, CA.

Then by ſimilar Triangles, $P : D :: an : am ::$ centripetal Force tending to C, : to centripetal Force tending to S :: verſed Sine of the Arch Aa : verſed Sine of the Arch (deſcribed in the ſame Time) whoſe Radius is $SA :: \frac{cc}{R} : \frac{ee}{D}$. Therefore $PR : DD :: cc : ee$.

Or thus, by Prob. V. Sect. II. $R = \frac{D\dot{D}}{\dot{P}}$; therefore $cc : ee :: P \times \frac{D\dot{D}}{\dot{P}} : DD$; that is $cc : ee :: P\dot{D} : D\dot{P}$.

Cor. 1. In the Ellipſis and Hyperbola, the Square of the Velocity of a Projectile moving round the Focus : is to the Square of the Velocity of a Body moving in a Circle at the ſame Diſtance :: as the Projectiles Diſtance from the other Focus : is to the Semi-tranſverſe.

For let $2r =$ Tranſverſe, $2b =$ Conjugate; then (by Ex. 3d. and 5th, Prob. XIX. Sect. II.) $P = \frac{bD}{\sqrt{2rD \mp DD}}$, FIG. 215. $\frac{bD}{\sqrt{2rD\mp DD}}$, and $\dot{P} = \frac{brD\dot{D}}{\overline{2rD\mp DD}|^{\frac{3}{2}}}$. Therefore $cc : ee :: \frac{bD\dot{D}}{\sqrt{2rD\mp DD}} : \frac{brD^2\dot{D}}{\overline{2rD\mp DD}|^{\frac{3}{2}}} :: 2r\mp D : r$.

Cor. 2. The Velocity of a Body revolving in an Ellipſis round the Center, is to the Velocity of a Body in a Circle at the ſame Diſtance; as the Conjugate to that Line of Diſtance, to the Diſtance itſelf.

For (by Ex. 4. Prob. XIX. Sect. II.) $P = \frac{br}{\sqrt{rr+bb-DD}}$, and $\dot{P} = \frac{brD\dot{D}}{\overline{rr+bb-DD}|^{\frac{3}{2}}}$, whence $cc : ee :: (\frac{br\dot{D}}{\sqrt{rr+bb-DD}} : \frac{brD^2\dot{D}}{\overline{rr+bb-DD}|^{\frac{3}{2}}} ::)\ rr+bb-DD : DD$. And $c : e ::$ Conjugate of D : to D.

Cor. 3. The Velocity of a Body moving in a Parabola about the Focus, is to the Velocity in a Circle at the ſame Diſtance : : as $\sqrt{2}$ to 1.

For let r = Latus Rectum, then by the Nature of the Parabola, $P = \frac{\sqrt{rD}}{2}$, and $\dot{P} = \frac{\dot{D}}{4}\sqrt{\frac{r}{D}}$. Whence $cc : ee :: (\frac{\dot{D}\sqrt{rD}}{2} : \frac{D\dot{D}}{4}\sqrt{\frac{r}{D}} ::)\ 2 : 1$.

PROB. IV.

FIG. 216.

To find the Velocity of a deſcending Body in any Place P, let fall from the given Point D towards the Earth or any attracting Body; being acted upon by a Force which is as any Power of its Diſtance CP from the Center.

Let the Earth's Radius $CA = r$, $CD = a$, $CP = x$, $DP = a - x$, whoſe Fluxion is $-\dot{x}$, $t =$ Time of deſcending thro' DP, $v =$ Velocity acquired by that Deſcent. F the Force at P, which let be as x^n.

By Mechanics, when the Body is given, it is univerſally $v\dot{v} \propto -F\dot{x} \propto -x^n\dot{x}$; (ſee Prob. I.) and the Fluent $v^2 \propto \frac{-x^{n+1}}{n+1}$. But in D, $v = 0$, $x = a$, therefore the Fluent corrected is $v^2 \propto \frac{a^{n+1} - x^{n+1}}{n+1}$.

But if $n = -1$, then $vv \propto \text{Log.} \frac{a}{x}$.

Now we muſt find the Value of v at A the Earth's Surface for ſome determinate Values of a and x, in order to turn the general Proportion into an Equation. Thus, it is known by Experiments, that a heavy Body deſcending through a Space s or $16\frac{1}{12}$ Feet will acquire a Velocity of $2s$ or $32\frac{1}{6}$ Feet in a Second of Time. Therefore writing $2s$ for v, r for x, $r+s$ for a, we ſhall get this Analogy, $\frac{\overline{r+s}^{n+1} - r^{n+1}}{n+1}$:

$$4ss :: \frac{a^{n+1} - x^{n+1}}{n+1} : vv = 4ss \times \frac{a^{n+1} - x^{n+1}}{\overline{r+s}^{n+1} - r^{n+1}}$$

$=$

FIG. 216. $=$ (becauſe $\overline{r+s}^{n+1}-r^{n+1}=\overline{n+1}\times r^{n}s$ nearly,) $4ss\times\frac{a^{n+1}-x^{n+1}}{\overline{n+1}\times r^{n}s}$; and $v=2s\sqrt{\frac{a^{n+1}-x^{n+1}}{\overline{n+1}\times r^{n}s}}$, the Feet deſcribed in a Second.

COR. 1. If $n=-2$, then $v=2r\sqrt{\frac{s\times\overline{a-x}}{ax}}$, therefore the Velocity of a Body falling from an infinite Diſtance to the Surface of the Earth will be $2\sqrt{rs}$.

COR. 2. If $n=0$, then $v=2\sqrt{s\times\overline{a-x}}$.

COR. 3. If $n=1$, then $v=\sqrt{\frac{2s}{r}\times\overline{aa-xx}}$. Therefore a Body falling from the Surface of the Earth to the Center acquires the Velocity $\sqrt{2rs}$.

COR. 4. *If the Body had been projected downwards from D with any Velocity c*, then inſtead of $v=0$, at D, put $v=c$, and the corrected Fluent will be $vv-cc\propto\frac{a^{n+1}-x^{n+1}}{n+1}$, or $vv-cc=4s\times\frac{a^{n+1}-x^{n+1}}{\overline{n+1}\times r^{n}}$. *Or if the Body was projected upwards*, $cc-vv\propto\frac{x^{n+1}-a^{n+1}}{n+1}$, or $cc-vv=4s\times\frac{x^{n+1}-a^{n+1}}{\overline{n+1}\times r^{n}}$, and in either Caſe $vv=cc+4s\times\frac{a^{n+1}-x^{n+1}}{\overline{n+1}\times r^{n}}$.

COR. 5. *If n be equal or greater than* -1; *as* -1, 0, 1, 2, *&c. a Body falling from an infinite Diſtance, will acquire an infinite Velocity:* For then a will be infinite, and a^{n+1} will be in the Numerator. But if n be leſs than -1; as -2, -3, -4, *&c.* the Velocity acquired at the Surface, in falling from an infinite Height, will be $\sqrt{\frac{-4sr}{n+1}}$, for then a is in the Denominator and infinite, and x is $=r$.

PROB.

PROB. V.

To find the Time wherein a falling Body will deſcend through any Space towards the Earth, &c. being acted upon by a Force which is as any Power of its Diſtance from the Earth's Center. FIG. 216.

The ſame Things ſuppoſed as in the laſt Problem, we ſhall have, the Moment of Time $\propto \frac{\text{Mom. Space}}{\text{Velocity}}$, univerſally. Since a deſcending Body at the Earth's Surface acquires the Velocity $2s$ in the Time p or 1 Second, therefore by the Laws of uniform Motion, $2s : p :: \dot{x} : \frac{p\dot{x}}{2s}$ = Moment of Time, wherein $\dot{x}$ is deſcribed with Velocity $2s$. Hence from the univerſal Proportion, $\frac{p\dot{x}}{2s} : \frac{\dot{x}}{2s} :: \dot{t} : \frac{-\dot{x}}{v}$; therefore $\dot{t} = \frac{-p\dot{x}}{v} = \frac{-\dot{x}}{v}$ (becauſe $p=1$). And $\dot{t} = \frac{-\dot{x}}{v} =$ (by the laſt Problem) $\frac{-\dot{x}}{2s\sqrt{\frac{a^{n+1}-x^{n+1}}{\overline{n+1}\times r^n s}}}$; whence $t =$ Fluent of $\frac{-\dot{x}}{2s\sqrt{\frac{a^{n+1}-x^{n+1}}{\overline{n+1}\times sr^n}}}$, expreſſed in Seconds.

Cor.

FIG. 216. Cor. 1. If $n=-2$, $t=$Fluent of $-\frac{1}{2r}\sqrt{\frac{a}{s}}\times\frac{x^{\frac{1}{2}}\dot{x}}{\sqrt{a-x}}$ $=$(by Form 10 and 11) $\frac{1}{2r}\sqrt{\frac{a}{s}\times\overline{ax-xx}}-\frac{a}{2r}\sqrt{\frac{a}{s}}$ $\times$,017453 $\times$ Degrees in the Arch whoſe Sine is $\sqrt{\frac{x}{a}}$, and Radius 1. And when duly corrected, the Time $t=\sqrt{\frac{aax-axx}{4rrs}}+\frac{,017453a}{2r}\sqrt{\frac{a}{s}}\times$ Degrees in the Arch whoſe Coſine is $\sqrt{\frac{x}{a}}$, and Radius 1. And the Time of deſcending to the Center is $\frac{3.1416a}{4r}\sqrt{\frac{a}{s}}$.

Cor. 2. If $n=0$, then $\dot{t}=\frac{-\dot{x}}{\sqrt{4s\times\overline{a-x}}}$, and $t=\sqrt{\frac{a-x}{s}}$.

Cor. 3. If $n=1$, then $\dot{t}=\sqrt{\frac{r}{2s}}\times\frac{-\dot{x}}{\sqrt{aa-xx}}$, and (by Form 10) $t=-\sqrt{\frac{r}{2s}}\times$,017453 Degrees of the Arch whoſe Sine is $\frac{x}{a}$, and Radius 1. And being duly corrected, $t=$,017453 $\sqrt{\frac{r}{2s}}\times$ Number of Degrees in the Arch whoſe Coſine is $\frac{x}{a}$, and Radius 1. Hence the Time of deſcending to the Center will be $\frac{3.1416}{2}\sqrt{\frac{r}{2s}}$: And therefore all the Times of Deſcent from any Altitudes whatſoever will be equal.

SCHOLIUM.

FIG. 216.

SCHOLIUM.

If t be given to find x; find the Fluent of $\dot t = -\frac{1}{2}r\sqrt{\frac{a}{s}} \times \frac{x^{\frac{1}{2}}\dot x}{\sqrt{a-x}}$, or $\sqrt{\frac{2r}{s}} \times \frac{-\dot x}{\sqrt{aa-xx}}$, by infinite Series and revert the Series. And if either t or v be given the other may be found by firſt finding x. And hence a Body being projected upwards or downwards with any Velocity; its Height may be found, and the Time of its Aſcent, or Deſcent. For then $\dot t = \frac{\dot x}{v} = \frac{\dot x}{\sqrt{cc+4s \times \frac{a^{n+1}-x^{n+1}}{\overline{n+1} \times r^{n}}}}$; by Cor. 4. Pr. 4:

PROB. VI.

The Velocity and Direction of a Projectile, and the Law of centripetal Force being given; to find the Velocities, Times and Angles of Revolution.

217. Let C be the Center of Force, and let the Body be projected from V in Direction VA with Velocity b deſcribing the Space h in the Time g; and let p be the Velocity and q the Space which the Force at V will generate in the ſame Time g. To the Center C deſcribe the Circle VXH, draw the Radii CX, CY infinitely near, cutting the Trajectory in I and K, and deſcribe the Arch Kr. Call CV, a; CI, x; VI, u; VX, z; Ir, $\acute{x}$; Kr, $\acute{y}$; YX, $\acute{z}$; IK, $\acute{u}$: And let $t =$ Time of deſcribing VI, and $\acute{t}$ of deſcribing IK, $s =$ Sine, $c =$ Coſine of the Angle CVA, and let the

FIG. 217. Force in any Place *I* be as x^n, and $v =$ Velocity in *I*. Then.

1. By the Reſolution of Forces, the Force to accelerate the Body in Direction of the Curve is $x^n \times \frac{Ir}{IK} = x^n \times \frac{-\acute{x}}{\acute{u}}$; but by Mechanics (ſee Prob. I.) $v\acute{v} \propto$ Force $\times \acute{u}$, univerſally. Whence $v\acute{v} \propto \frac{-x^n\acute{x}}{\acute{u}} \times \acute{u} \propto -x^n\acute{x}$. Hence therefore the Moment of Velocity depends not at all on the Angle *KIC*, but upon $\acute{x}$ the Moment of perpendicular Deſcent, and is therefore the ſame at all Inclinations as if the Body deſcended perpendicularly. Now to find the Moment of Velocity in *V*; by the Laws of uniformly accelerated Motion, the Velocity generated is as the Time, or as the Space uniformly deſcribed with a given Velocity: And ſince in the Time of deſcribing $2q$, the Velocity p is generated, therefore $2q : p : : \acute{x} : \frac{p\acute{x}}{2q}$ = Moment of Velocity generated at *V* whilſt $\acute{x}$ is deſcribed with Velocity p. Here therefore the Value of $v\acute{v}$ is $\frac{pp\acute{x}}{2q}$, and Force $\times$ Moment of Space is $a^n\acute{x}$. Therefore from the univerſal Proportion, $\frac{pp\acute{x}}{2q} : a^n\acute{x} : : v\acute{v} : -x^n\acute{x} : : v\dot{v} : -x^n\dot{x}$, whence $v\dot{v} = \frac{-ppx^n\dot{x}}{2qa^n}$, and the Fluent $v^2 = \frac{-ppx^{n+1}}{\overline{n+1} \times qa^n}$. But in *V*, $x=a$, and $v=b$, therefore by Correction $vv = bb + \frac{ppa}{\overline{n+1}.q}$

FIG. 217.

$-\frac{ppx^{n+1}}{\overline{n+1}.qa^{n}}$. But here if $n=-1$, then $vv=bb+\frac{ppa}{q}\times\text{Log.}\frac{a}{x}$.

2. Again, ſince the Velocity is every where reciprocally as the Perpendicular let fall on the Tangent; and theſe Perpendiculars in *V* and *I* are sa, and $\frac{x\acute{y}}{\acute{u}}$; therefore $b:v::\frac{x\acute{y}}{\acute{u}}:sa$, then is $\acute{y}=\frac{asb}{vx}\acute{u}=\frac{asb}{vx}\sqrt{\acute{x}^{2}+\acute{y}^{2}}$, and by Reduction $\acute{y}=\frac{-asb\acute{x}}{\sqrt{v^{2}x^{2}-a^{2}s^{2}b^{2}}}$: But by ſimilar Triangles $x:\acute{y}::a:\acute{z}=\frac{a\acute{y}}{x}$, and $\dot{z}=\frac{a\dot{y}}{x}$, therefore $\dot{z}=\frac{-aasb\dot{x}}{x\sqrt{v^{2}x^{2}-a^{2}s^{2}b^{2}}}$.

3. Laſtly, ſince $\acute{t}\propto\frac{x\acute{y}}{2}$ the Moment of the Area, or the Time $\propto$ Area; and in *V*, the Area $=\frac{asb}{2}$; therefore as $g:asb::\acute{t}:x\acute{y}::\dot{t}:x\dot{y}$, then will $\dot{t}=\frac{gx\dot{y}}{asb}$, or $\dot{t}=\frac{-gbx\dot{x}}{b\sqrt{v^{2}x^{2}-a^{2}s^{2}b^{2}}}$.

Conſequently ſubſtituting for vv it's equal, in the Values of $\dot{z}$ and $\dot{t}$, the Fluents will give z and t.

Cor. 1. Hence the Apſides of the Trajectory are eaſily found; for then it will be $\acute{u}=\acute{y}=\frac{asb}{vx}\acute{u}$, or $asb=vx$; therefore $vv=\frac{a^{2}s^{2}b^{2}}{xx}=bb+\frac{ppa}{\overline{n+1}\times q}-\frac{ppx^{n+1}}{\overline{n+1}\times qa^{n}}$. Whence x will be had: And if

FIG. 217. two Roots of this Equation be found; then the correſpondent Fluents z will give the Poſition of the Apſides.

Cor. 2. Since the Sine of the Angle $CIK = \frac{\dot{y}}{\dot{u}} = \frac{asb}{vx}$; therefore if that Angle be given, the Diſtance x may be found; or if x be given, the Angle may be found.

Cor. 3. If the Body be projected at right Angles to CV, then $s = 1$, and, by Cor. 1, we ſhall have $bbxx + \frac{ppax^2}{\overline{n+1} \times q} - \frac{ppx^{n+3}}{\overline{n+1} \times qa^n} = a^2bb$; in which one Root is a; and finding x another Root in the Equation, the Fluent z may thence be had; and conſequently the Motion of the Apſides.

Cor. 4. If in any Part of the Orbit, $vx = asb$, and v be greater than $\sqrt{\frac{ppx^{n+1}}{2qa^n}}$, the Body will fly off from the Center; but if v be leſs, it will approach the Center; if equal, it will move in a Circle.

For by the Laws of centripetal Force $\sqrt{2aq}$ will be the Arch deſcribed in the Time g, by a Body revolving in a Circle at the Diſtance a; and $2q : p :: \sqrt{2aq} : \sqrt{\frac{ppa}{2q}} =$ the Velocity of a Body moving in a Circle at the Diſtance a. And $\sqrt{a^{n+1}} : \sqrt{x^{n+1}} :: \sqrt{\frac{ppa}{2q}} : \sqrt{\frac{ppx^{n+1}}{2qa^n}} =$ the Velocity of a Body moving in a Circle at the Diſtance x. Conſequently when v is greater, leſſer, or equal to $\sqrt{\frac{ppx^{n+1}}{2qa^n}}$, the Body will fly from, or approach the Center, or move in a Circle.

PROB.

PROB. VII.

To find the Time of a Body's deſcending thro' any Arch of a Cycloid.

Let *AC* be the Axis, *BV*, *DF*, *ef*, Ordinates. Let $AC = a$, $VC = b$, $VF = x$, $Ff = \dot{x}$, $BD = z$, $De = \dot{z}$, $s = 16\frac{1}{12}$ Feet. t = Time of deſcribing BD: And let the Body fall from B. FIG. 218.

The Times of deſcribing any Spaces uniformly are as the Spaces directly, and the Velocities reciprocally; but the Velocities are as the Square Roots of the Heights fallen from; and $2s$ is the Space uniformly deſcribed in 1 Second, by the Velocity acquired by falling through s: Therefore $\frac{2s}{\sqrt{s}}$: 1 Second :: $\frac{\dot{z}}{\sqrt{x}} : \dot{t} = \frac{\dot{z}}{2\sqrt{sx}}$; or $\dot{t} = \frac{\dot{z}}{2\sqrt{sx}}$; but (by Ex. 8. Prob. VIII. Sect. II.) $\dot{z} = \dot{x}\sqrt{\frac{a}{b-x}}$; whence $\dot{t} = \frac{1}{2}\sqrt{\frac{a}{s}} \times \frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{b-x}}$. Whence the Fluent (by Form the 10th) is $t = \sqrt{\frac{a}{s}} \times$ Arch whoſe Sine is $\sqrt{\frac{x}{b}}$, and Radius 1. And when $x = b$, then $t = \frac{3.1416}{2}\sqrt{\frac{a}{s}}$. Whence if a Pendulum be made to vibrate in the Arch of the Cycloid, $2t$ or the Time of one entire Vibration will be $3.1416\sqrt{\frac{a}{s}}$, in Seconds.

COR.

FIG. 218. COR. 1. Hence all the Times are equal in which Bodies deſcending from any Points Z, B, D ſhall arrive at the loweſt Point C: And all the Times of Vibration will be equal among themſelves.

COR. 2. It appears by Ex. 3. Prob. VII. Sect. II. that if ZP, PQ be two Cycloids, whoſe Cuſpids are at P, and Vertices at Z and Q; then if a Pendulum PC be ſuſpended at P, ſo that in oſcillating it may fold about the Curves ZP, PQ; then the Point C will deſcribe the Cycloid ZCQ: And therefore the Time of its Vibration will be $3.1416 \times$ Time of a Body's falling through AC.

PROB. VIII.

219. *To find the Force wherewith a Corpuſcle P is attracted to the Plane of a Circle EQD, according to any Law of centripetal Force.*

Let A be the Center, and AP perpendicular to the Plane of the Circle, and let the Force of each Particle be as the n^{th} Power of the Diſtance.

Put $AP = a$, $AE = x$, $c = 3.1416$. And let the Body m attract the Corpuſcle P, at the Diſtance d, with the Force f; then the Force which any Particle $\dot{x}$ attracts the Corpuſcle P towards E is $\frac{f\dot{x}}{md^n} \times \overline{aa+xx}^{\frac{n}{2}}$, and the Force of all the Particles in the Periphery EQD is $\frac{2cfx\dot{x}}{md^n} \times \overline{aa+xx}^{\frac{n}{2}}$: And by Mechanics the Force in Direction $PA = \frac{2cfax\dot{x}}{md^n} \times$

$aa+xx$

$\overline{aa+xx}^{\frac{n-1}{2}}$; and the Fluxion of the Force $= \frac{2cafxx}{md^n}$ FIG. 219.
$\times\, \overline{aa+xx}^{\frac{n-1}{2}}$; whoſe Fluent is $\frac{2cfa}{md^n} \times \frac{\overline{aa+xx}^{\frac{n+1}{2}}}{n+1}$. But in *A*, $x=0$; therefore by Correction, the Force exerted on *P* by the Plane of the Circle *ED* is $= \frac{2cfa}{md^n} \times \frac{\overline{aa+xx}^{\frac{n+1}{2}}}{n+1} - \frac{2cfa^{n+2}}{\overline{n+1} \times md^n}$. But if $n = -1$, then the Force $= \frac{cfad}{m} \times 2.302585$ Log. $\frac{aa+xx}{aa}$.

Cor. 1. Hence if $n=1$, the Force of the Circle exerted on the Particle *P* will be $\frac{cfaxx}{md}$, the ſame as if the ſaid Circle were wholly collected into the Center *A*.

Cor. 2. Therefore if $n=1$, a Sphere will attract any Particle *P* with the ſame Force as if the whole Sphere was contracted into the Center *C*. For taking the Circles *ED*, *ed* parallel, and equidiſtant from the Center; the Sum of the Forces will be as the Sum of the two Circles each multiplied into its Diſtance, or as either Circle into half the Sum of the Diſtances, that is into $2PC$; or both Circles multiplied into *PC*; and it is the ſame of all equidiſtant Circles that compoſe the Globe. 220.

Cor. 3. If $n=-2$, the Force $= \frac{2cfdd}{m} \times \overline{1 - \frac{a}{\sqrt{aa+xx}}}$ 219.
$= \frac{2cfdd}{m} \times \overline{1 - \frac{PA}{PE}}$.

Cor. 4. If *n* be leſs than -1, then the Force of the whole infinite Plane will be $\frac{2cfa^{n+2}}{md^n \times \overline{-n-1}}$.

PROB.

PROB. IX.

FIG. 221. *To find the Force wherewith an infinite Solid, plain on one Side L l, attracts a Corpuſcle placed at C: Suppoſing the Law of Attraction to be univerſally as ſome Power of the Diſtance greater than* 1.

Draw the infinite Line *CGK*, perpendicular to the Plane *LGl*, and through the Points *I*, *K*, infinitely near each other, draw two Planes parallel to *L l*; and let $CG = a$, $CI = x$, $IK = \acute{x}$, and the Force as x^n; and by Cor. 4. Prob. VIII. the Force wherewith the Solid contained between the Planes at *I*, *K*, attracts the Corpuſcle, will be $\frac{2cfx^{n+2}\acute{x}}{md^n \times \overline{-n-1}}$, and its Fluxion $\frac{2cfx^{n+2}\dot{x}}{md^n \times \overline{-n-1}}$, whoſe Fluent is $\frac{2cf}{-md^n} \times \frac{x^{n+3}}{\overline{n+1} \times \overline{n+3}}$. And when duly corrected the Force will be $\frac{2cf}{\overline{n+1} \times \overline{n+3} \times md^n} \times \overline{a^{n+3} - x^{n+3}}$.

COR. 1. If n be leſs than -3, and the Solid infinite towards *K*; the Force will be $\frac{2cfa^{n+3}}{\overline{n+1} \times \overline{n+3} \times md^n}$.

COR. 2. Hence therefore the Force at different Diſtances from the infinite Solid (when n is leſs than -3) will be as a^{n+3} or CG^{n+3}.

COR. 3. Hence alſo (if n be leſs than -3), the Force of a very great Body upon a very ſmall Particle, at any extremely ſmall Diſtances, will be as the $\overline{n+3}^{th}$ Power of the Diſtance, nearly. FIG. 221.

Cor. 4. And if the Corpuſcle be placed within the Solid at *H*, ſo as $GH = GC$, the Force will be the very ſame as if it were placed at *C*, ſo far without it. For, taking $IH = GH$, the Solids *HG* and *HI* deſtroy one another's Effects.

PROB. X.

To find the Force wherewith a Sphere attracts a Corpuſcle P, ſituated either without or within the Sphere; ſuppoſing the Forces of all the Particles to be reciprocally as the Squares of their Diſtances.

Caſe 1. Let *P* be without the Sphere. Draw the Axis *PAB*, and the Ordinates *ED*, *ed* infinitely near, and let *S* be the Center, put $PS = a$, $PD = x$, $Dd = \dot{x}$, $AS = r$, $ED = y$, $bb = aa - rr$. Then $PE = \sqrt{xx+yy} = \sqrt{xx+rr-\overline{a-x}^2} = \sqrt{rr-aa+2ax} = \sqrt{2ax-bb}$. But by Cor. 3. Prob. VIII. the Force of the infinitely thin Solid contained between the Planes of the Circles *DE*, *de* is $= \frac{2fcdd\dot{x}}{m} \times \overline{1 - \frac{PD}{PE}}$, herefore the Fluxion of the Force is $\frac{2cfdd}{m} \times \overline{\dot{x} - \frac{x\dot{x}}{\sqrt{2ax-bb}}}$, whoſe Fluent is $\frac{2cfdd}{m} \times \overline{x - \frac{bb+ax}{3aa}\sqrt{2ax-bb}}$, but in *A* where the Force s 0, $x = a - r$; therefore by proper Correction, the 222.

Force of the Segment EAF is $=\frac{2cfdd}{m}\times$ into: $x-a+r+\frac{bb+aa-ar}{3aa}\times\overline{a-r}-\frac{bb+ax}{3aa}\times PE$: and when $x=a+r$, the Force of the whole Sphere is $=\frac{2cfdd}{m}\times\frac{2r^3}{3aa}$.

223. *Caſe* 2. Let P be within the Sphere. Let $PQ=b$, $PD=x$, then $PE=\sqrt{rr-aa+2ax}=\sqrt{bb+2ax}$; therefore (by Cor. 3. Pr. VIII.) the Fluxion of the Force at D is $\frac{2cfdd}{m}\times\dot{x}-\frac{x\dot{x}}{\sqrt{bb+2ax}}$; whoſe Fluent is $\frac{2cfdd}{m}\times : x+\frac{bb-ax}{3aa}\sqrt{bb+2ax}$: but in P, $x=0$; therefore the Force of the Zone or Section $QEDFR=x+\frac{bb-ax}{3aa}\sqrt{bb+2ax}-\frac{b^3}{3aa}$. In which writing $a+r$ for x, there comes out $\frac{2cfdd}{m}\times\frac{r^3+a^3-b^3}{3aa}$ for the Attraction of the Segment QBR. And by a like Proceſs the Attraction of the Segment $QAR=\frac{2cfdd}{m}\times\frac{r^3-a^3-b^3}{3aa}$: whoſe Difference is $\frac{2cfdd}{m}\times\frac{2}{3}a$, the abſolute Force of the Corpuſcle P towards the Center; which is the ſame as the Force of a Sphere, whoſe Radius is SP, acting on the Corpuſcle P at its Surface.

Cor. 1. Hence the Force of the Sphere upon the Particle P placed without the Sphere is the very ſame as if the whole Sphere was collected into the Center, and exerted the Sum of all the Forces from that Center. For $\frac{4r^3c}{3aa}=$ Sphere divided by the Square of the

FIG. 223.

the Diſtance; and $\frac{m}{dd} : f :: \frac{4r^3c}{3aa} : \frac{4cr^3d^2f}{3ma^2}$ the very ſame Force of the Sphere before found. And hence it is alſo evident, that the Forces of Spheres are accurately in the reciprocal Ratio of the Squares of the Diſtances from their Centers.

Cor. 2. The Force wherewith any Corpuſcle P within a Sphere, is attracted to the Center, is accurately as its Diſtance from the Center; and is the ſame in different homogeneous Spheres, as long as that Diſtance is the ſame.

Cor. 3. Hence alſo, the Forces of Gravity at the Surfaces of any homogeneous Spheres, are as the Radii of the Spheres. For the Force is as $\frac{2cfdd}{m} \times \frac{2r^3}{3aa}$, that is as $\frac{r^3}{aa}$, and when $a = r$, it will be as $\frac{r^3}{rr}$ or as r.

PROB. XI.

To find the Force wherewith a Spheroid attracts a Corpuſcle P, lying upon its Surface in the Axis PB. 224.

Let $PB = 2r$, Diameter $GH = 2a$, $AP = x$, then $AE = \frac{a}{r}\sqrt{2rx - xx}$, and $PE = \sqrt{xx + \frac{aa}{rr} \times \overline{2rx - xx}}$ $= \frac{1}{r}\sqrt{2ra^2x + rrxx - aaxx} = \frac{1}{r}\sqrt{2raax + bbxx}$, putting $bb = rr - aa$. By Cor. 3. Prob. VIII. the Fluxion of the Force at A is $= \frac{2cfdd\dot{x}}{m} \times \overline{1 - \frac{PA}{PE}}$

=

FIG. 224. $= \frac{2cfdd}{m} \times : \overline{\dot{x} - \frac{rx^{\frac{1}{2}}\dot{x}}{\sqrt{2ra^2+bbx}}}$: Let $\varphi = \frac{2.302585}{\frac{1}{2}b}$ $\times$ Log. $\frac{b\sqrt{x}+\sqrt{2raa+bbx}}{\sqrt{2raa}}$, if r be greater than a: or $\varphi = \frac{,017453}{\frac{1}{2}b} \times$ Degrees in the Arch, whoſe Sine is $\frac{b}{a}\sqrt{\frac{x}{2r}}$, if r is leſs than a; and the Fluent is $= \frac{2cfdd}{m} \times : x + \frac{rraa}{bb}\varphi - \frac{r}{bb}\sqrt{2ra^2x+b^2x^2}$: = Force wherewith the Segment PEF attracts the Particle at P.

COR. 1. *Hence the Force wherewith the whole Spheroid attracts the Particle P is* $= \frac{2cddf}{m} \times$ *into* $2r + \frac{r^2a^2}{bb}\varphi - \frac{2r^3}{bb}$: *where* $\varphi = \frac{2.302585}{\sqrt{rr-aa}} \times$ 2 *Log*: $\frac{r+\sqrt{rr-aa}}{a}$: *or* $\varphi = \frac{,017453}{\sqrt{aa-rr}} \times$ *twice the Number of Degrees in the Arch whoſe Sine is* $\frac{\sqrt{aa-rr}}{a}$, *Radius* = 1, *according as r is greater or leſſer than a.*

COR. 2. *If b be very ſmall, the Force of the Segment EPF* $= \frac{2cdd}{m} f \times$ *into* $x - \frac{2rx}{3a}\sqrt{\frac{x}{2r}} + \frac{bbxx}{10a^3}\sqrt{\frac{x}{2r}}$: *very near. And the Force of the whole Spheroid is* $= \frac{2cfdd}{m} \times$ into $2r - \frac{4rr}{3a} + \frac{2rr}{5a^3}bb$; as will appear by infinite Series, or Form the 16th.

COR.

FIG. 227. Cor. 3. Let $APBQ$ be an Ellipſis, AB the greater Axis, PQ the leſſer. Put $\varphi = \dfrac{2.30258}{\sqrt{AC^2 - PC^2}} \times$ Log: $\dfrac{AC + \sqrt{AC^2 - PC^2}}{PC}$: then the attracting Force at A upon the oblong Spheroid, generated by revolving round AB, is $2AC + \dfrac{2AC^2 \times PC^2}{AC^2 - PC^2}\varphi - \dfrac{2AC^3}{AC^2 - PC^2}$. And (by the laſt Prob.) the Force at A upon a Sphere whoſe Diameter is AB, is $\frac{2}{3}AC$. *Therefore the Force of the Spheroid, to that of the Sphere, is as* $\overline{AC \times \varphi - 1} \times 3PC^2$, *to* $AC^2 - PC^2$. And ſince the oblate Spheroid generated by revolving about PQ, is a mean proportional between the oblong Spheroid and this Sphere; *therefore the Force upon the Equinoctial at A, upon this oblate Spheroid* (which repreſents the Earth) *is nearly a mean between the Forces of theſe other two Bodies.* And this may help to determine the Figure of the Earth.

PROB. XII.

To find the Motion of a Ray of Light paſſing into a refracting Medium.

Let there be two Mediums ſeparated by the refracting Space $RAdD$ terminated by the Parallel Planes RA, Dd; and let the Ray, moving in the Direction GH, paſs from H to I, and in its Paſſage be acted upon, in Lines perpendicular to the Planes, by any Force which is equal at equal Diſtances from either Plane, and at all Diſtances as any Powers or Sums of Powers of the Diſtance therefrom. And let 225.

FIG. 225. let b = Velocity in H, v = Velocity in P, $CP = x$, $PI = z$; $PF = \acute{z}$, $\acute{t}$ = Time of deſcribing PF. And let the Force be as $A + Bx^m + Cx^n + Dx^r$ &c. $= \mathcal{Q}$; then will the Force in Direction PF be $\frac{\acute{x}}{\acute{z}}\mathcal{Q}$. And ſince the Velocity $\propto \frac{\text{Space}}{\text{Time}}$, and Moment of Velocity $\propto$ Force × Time, therefore $v\acute{v} \propto \mathcal{Q}\acute{x}$, or $v\dot{v} \propto \mathcal{Q}\dot{x}$; aſſume p a given Quantity, and let $v\dot{v} = p\mathcal{Q}\dot{x}$, and let F be the Fluent of $\mathcal{Q}\dot{x}$, then $v^2 = 2pF$, and by Correction $vv - bb = 2pF$, and $vv = bb + 2pF$. Hence the Ray will always have the ſame Velocity in the ſame Medium DIK, whatever be the Angle of Incidence.

Let the Motion of the Ray GH be divided into two GA, AH, one parallel the other perpendicular to the Plane RA. Then ſince the parallel Motion AH is not at all changed by the Actions of the Forces perpendicular to theſe Planes: Therefore if ID be made $= AH$, and DR perpendicular to DI, then IK will be deſcribed in the ſame Time as GH. Therefore drawing IE parrallel to GH, the Velocity in H to the Velocity in I, is as GH or EI to IK, that is as the Sine of the Angle of Refraction to the Sine of the Angle of Incidence. And therefore the Velocity of Light in Vacuo, to its Velocity in Air of a mean Denſity at the Surface of the Earth; is as ,$999\frac{3}{4}$ to 1. For by Experiment the Sines of Refraction and Incidence are in that Ratio.

COR. 1. *The Sine of the Angle of Incidence at one Plane, is to the Sine of the Angle of Emergence from the other Plane, in a given Ratio.* For the Velocity v at the ſecond Plane will always be equal to the given Quantity $\sqrt{bb + 2pF}$, and the Sine of Incidence to the Sine of Emergence, as this given Quantity to b.

Cor. 2. *If the Ray was to fall on I in Direction KI, with the Velocity it has at I, it would return in the ſame Curve IPH, and ſo go to G, and obtain its firſt Velocity.* For the ſame Forces that did before accelerate its Paſſage, will now equally retard it in returning. Therefore,

Cor. 3. *If the Ray have a greater Velocity in the firſt Medium, than in the ſecond, and the Angle of Incidence GHA be continually diminiſhed, the Ray will at laſt be reflected; and the Angle of Reflection gha will be equal to the Angle of Incidence GHA.* 226.

For let the Angle *GHA* be ſuch, that the Ratio of its Coſine to the Radius may be equal or greater than the Ratio of the Sine of Incidence of the firſt Medium, to the Sine of Emergence in the ſecond; and the Ray at *R* will be moving in a Direction parallel to the Planes; but being afterwards acted on by the ſame Forces as before, it will be turned back deſcribing the Line *Rhg* ſimilar and equal to *RHG*, and the Angle $gha = GHA$.

Cor. 4. *Hence if there be two ſimilar Mediums whoſe Denſities are p and q; and the Velocities after Refraction into each of them z and y: Then will* $zz - bb : yy - bb :: p : q$.

For ſince the Forces of Attraction are made towards Bodies, theſe Forces will be proportional to the Cauſes that produce them, and therefore will be as the Denſities of theſe Bodies, ſuppoſing the internal Form and Conſtitution of the Bodies to be in other Reſpects the ſame. The Forces therefore exerted at any equal Diſtances by theſe two Mediums will be as pQ and qQ; whence will be had $zz - bb = 2pF$, and $yy - bb = 2qF$; whence $zz - bb : yy - bb :: p : q$, that is as the Denſities of the Bodies, nearly.

Cor. 5. *If Light paſs thro' ſeveral refracting Mediums, the Sum of all the Refractions will be equal to the ſingle Refraction it would have ſuffered, by paſſing immediately out of the firſt Medium into the laſt.*

FIG. 226. For ſuppoſing theſe ſeveral Mediums to be ſeparated by parallel Planes, the Refraction, Velocity, or Motion generated in approaching any one of theſe Mediums, will be deſtroyed again in its receding (on the other Side) from the ſame Medium. And therefore the Motion of the Ray can only be affected with the Force of that Medium it at laſt moves in.

SCHOLIUM.

225. Though it is not known to what preciſe Diſtance the refractive Power of any Medium reaches; yet we are ſure it is contained in an exceeding ſmall Compaſs. And therefore the Curve *HPI*, and conſequently the Points *H*, *I* may be taken in Practice only as one Point.

There are few or no Examples among all the Phænomena of Nature that afford ſo clear a Proof of the prodigious Forces of the ſmall Particles of Matter, as the Motion and Refraction of Light does. For notwithſtanding the amazing Velocity of the Rays, and the extremely ſmall Space and Time that any refracting Surface has to act in; and yet to produce ſuch a ſenſible Refraction as we ſee it does, muſt evince that the Forces exerted on theſe ſmall Bodies muſt be ſurpriſingly great, and do really exceed all Comprehenſion.

PROB. XIII.

The Velocity of a Globe, moving in a right Line, and its Denſity, and the Denſity of the reſiſting Medium in which it moves being given; to find the Time, Velocity, and Space deſcribed.

Here we ſuppoſe the Medium to be uniform, and that the Projectile is acted on by no Force but the Reſiſtance of the Medium, and that to be as the Square of the Velocity.

Let W = Weight of the Globe, d = the Diameter, q = it's Denſity, p = Denſity of the Medium, s = a Space of 16$\frac{1}{12}$ Feet, b = Space the Globe at firſt can deſcribe in 1 Second, or the firſt Velocity; x = any Space deſcribed, t = Time of deſcribing it, and v = Velocity at the End of that Time. Here I meaſure the Velocity by the Space deſcribed in a Second, and the Time is Seconds.

1. It is proved by Experiments that the Reſiſtance of the Globe is to the Force by which its Motion may be generated in the Time of deſcribing $\frac{8}{3}$ its Diameter, as the Denſity of the Fluid to the Denſity of the Globe nearly. The Velocity generated in a given Body is as the Force and Time conjunctly, therefore the Force is as the Velocity divided by the Times or by Spaces uniformly deſcribed in theſe Times, therefore $\frac{2s}{b} : W :: \frac{b}{\frac{8}{3}d} : \frac{3bbW}{16ds}$ = Force that will generate the Globes Motion in the Time it deſcribes $\frac{8}{3}d$; therefore $\frac{3bbp}{16dsq}W$ = Reſiſtance of the Globe

with Velocity b, and likewiſe $\frac{3pvv}{16dsq}W =$ Reſiſtance with the Velocity v. Now 1 Second : $2s$ (Velocity of a falling Body) : : $\dot{t} : 2s\dot{t} =$ Velocity generated in the Time $\dot{t}$ by Gravity. Now ſince the Moment of Velocity is, in all Caſes as the Force and Moment of Time; therefore $2s\dot{t} : W\dot{t} : : -\dot{v} : \frac{3pvv}{16dsq}W \times \dot{t}$; therefore $\dot{v} = \frac{-3pv^2\dot{t}}{8dq}$, or $v^{-2}\dot{v} = \frac{-3p\dot{t}}{8dq}$; whence the Fluent $v^{-1} = \frac{3pt}{8dq}$. But when $t = 0$, $v = b$, therefore the Fluent corrected is $\frac{1}{v} - \frac{1}{b} = \frac{3pt}{8dq}$, which reduced is $8dqv + 3pbtv = 8dqb$.

2. 1 Second : $2s$ (Space uniformly deſcribed with Velocity $2s$) : : $\dot{t} : 2s\dot{t} =$ Moment of Space deſcribed with Velocity $2s$. And ſince Moment of Velocity $\propto \frac{\text{Force}}{\text{Velocity}} \times$ Moment of Space; and the Moment of Velocity in falling Bodies was found before $= 2s\dot{t}$. Therefore $2s\dot{t} : \frac{W \times 2s\dot{t}}{2s} : : -\dot{v} : \frac{3pvvW}{16dsq} \times \frac{\dot{x}}{v}$; whence $-\dot{v} = \frac{3pv\dot{x}}{8dq}$, and $\frac{-\dot{v}}{v} = \frac{3p\dot{x}}{8dq}$. Therefore the Fluent is $\frac{3px}{8dq} = -2.3025 \times$ Log. v. And by Correction $\frac{3px}{8dq} = 2.302585$ Log: $\frac{b}{v}$:

COR. 1. *Hence* $x = \frac{8dq}{3p} \times 2.302585\,\text{Log}: \overline{1 + \frac{3pbt}{8dq}}:$ *this appears by expunging* v.

Cor. 2. *If* $T = \frac{8dq}{3bp}$, *then* $v = \frac{Tb}{T+t}$; *and* $x = bT \times 2.302585$ Log: $\frac{T+t}{T}$; *and Log:* $\frac{v}{b} = \frac{-x}{2.30258bT}$.

Cor. 3. *If the Reſiſtance was to be as any Power of the Velocity* v^n; *then by the ſame way of Reaſoning, we ſhould get* $\frac{3pv^nW}{16dsqb^{n-2}}$ = *Reſiſtance of the Globe with Velocity* v; and thence $\frac{-\dot v}{v^n} = \frac{3p\dot t}{8dqb^{n-2}}$; and the correct Fluent, $\frac{b^{1-n} - v^{1-n}}{1-n} = \frac{3pt}{8dqb^{n-2}}$: But when $n=1$, Log: $\frac{b}{v} = \frac{3pbt}{8dq}$.

Likewiſe $\frac{-\dot v}{v^{n-1}} = \frac{3p\dot x}{8dqb^{n-2}}$, *and the correct Fluent* $\frac{b^{2-n} - v^{2-n}}{2-n} = \frac{3px}{8dqb^{n-2}}$.

Cor. 4. *If it was required to find the Motion of any Sort of Body B moving in a right Line, and reſiſted as the* n^{th} *Power of the Velocity; it's done in the ſame Manner.* Let the Reſiſtance of B moving with Velocity 1, be = Weight of the Body w. Then $1 : w :: v^n : wv^n$ = the Reſiſtance it meets, with Velocity v. Then (ſee Prob. I.) $-v\dot v \propto \frac{F\dot s}{b}$, and in the Caſe of falling Bodies, $v=2b$, and $\dot v = \dot s$, $F = w = b$. And in the Caſe of Reſiſtance $F=wv^n$, $b=B$. Therefore $2b\dot s : \frac{w\dot s}{w} :: -v\dot v : \frac{wv^n\dot s}{B}$. Therefore $-v\dot v = \frac{2bwv^n\dot s}{B}$, FIG. $\frac{2bwv^{n}\dot{s}}{B}$, or $-v^{1-n}\dot{v} = \frac{2bw\dot{s}}{B}$; *and the Fluent is* $\frac{b^{2-n}-v^{2-n}}{2-n} = \frac{2bw}{B}s$.

Alſo $-\dot{v} \propto \frac{F\dot{t}}{b}$, and in the Caſe of falling Bodies $\dot{v} = 2b\dot{t}$, whence $2b\dot{t} : \frac{w\dot{t}}{w} :: -\dot{v} : \frac{wv^{n}\dot{s}}{B}$. And $v^{-n}\dot{v}, = \frac{2bw}{B}\dot{t}$, *and the Fluent is* $\frac{b^{1-n}-v^{1-n}}{1-n} = \frac{2bw}{B}t$.

And hence when $v=0$, *or all the Motion is deſtroyed, and* n *leſs than* 2, *then is* $\frac{b^{2-n}}{2-n} = \frac{2bw}{B}s$, *or if* n *be leſs than* 1, $\frac{b^{1-n}}{1-n} = \frac{2bw}{B}t$; *thence the whole Space and Time* s *and* t *will be known. But in all other Caſes* s *and* t *will be infinite, and* n *being leſs than* 2, *the whole Space* $s \propto b^{2-n}$, *for any the ſame Value of* n.

SCHOLIUM.

It is here laid down as a Principle that the Reſiſtance of a Globe moving in a reſiſting Medium, is to the Force by which its Motion may be generated in the Time of deſcribing $\frac{8}{3}$ its Diameter; as the Denſity of the Medium to the Denſity of the Globe: Yet I have found by ſome Experiments that in ſwift Motions, the Reſiſtance has been greater ſometimes by a third or fourth Part. Theſe Experiments I tried in a River, with a Globe of equal Denſity with the Water, by faſtning a Thread to the Globe and to an Index incloſed in a Tube with a ſpiral Spring: For by the Diviſions of the Index, as it was drawn out more or leſs, I could meaſure the Reſiſtance.

The Reaſon of this greater Reſiſtance ſeems to be, that the Globe being near the Surface of the Water, and the Motion of the Water being very ſwift, the Water had not Liberty to diverge upwards from the Globe, and upon this Account made the Reſiſtance greater than if the Globe had been deeply immerſed; for then the Water could have diverged freely from the Globe in all Directions, and on all Sides.

PROB. XIV.

If a Body in a uniform Medium, being uniformly acted on by the Force of Gravity, aſcends or deſcends in a right Line; To find the Times, Velocities, and Spaces deſcribed.

Suppoſe as before W = *Weight of the Globe.*
d = *its Diameter.*
q = *its Denſity.*
p = *Denſity of the Medium.*
$s = 16\frac{1}{12}$ *Feet the Space through which a heavy Body deſcends by Gravity in a Second.*
x = *Space deſcribed from the beginning of the Motion.*
t = *Time of deſcribing* x.
v = *Velocity at the End of the Time* t.
b = *Velocity the Body is projected upwards or downwards with. Here I meaſure the Velocity by the Space uniformly deſcribed in a Second.*

The comparative Weight of the Globe in the Medium will be $\frac{q-p}{q}W$. And we ſhall find, as in the laſt Problem $\frac{3pvv}{16dsq}W$ = Reſiſtance of the Globe moving with Velocity v; and conſequently $\frac{q-p}{q}W \pm \frac{3pvv}{16dsq}W$ is the Force acting on the Globe according as it aſcends or deſcends; call this Force y. Now $2s\acute{t}$ = Moment of Velocity generated by Gravity in the Time $\acute{t}$: And the Moment of Velocity being as the Force and Moment of Time univerſally; therefore $2s\acute{t} : W\acute{t} :: \mp\acute{v} : y\acute{t} :: \mp\dot{v} : y\dot{t}$; whence $\dot{t} = \mp\frac{W\dot{v}}{2sy} = \frac{\mp 8dq\dot{v}}{\overline{q-p}\times 16ds \pm 3pvv}$.

Again, let a falling Body deſcribe any ſmall Space $\acute{z}$ at the End of 1 Second; then it will be, $2s$ (Space uniformly deſcribed with Velocity $2s$) : 1 Second :: $\acute{z} : \frac{\acute{z}}{2s}$ = Time of deſcribing $\acute{z}$ with Velocity $2s$. And ſince the Moment of Space ∝ Velocity × Moment of Time univerſally: therefore $\acute{z} : 2s\times\frac{\acute{z}}{2s} :: \acute{x} : v\acute{t} :: \dot{x} : v\dot{t}$. Whence $\dot{x} = v\dot{t} = \frac{\mp 8dqv\dot{v}}{\overline{q-p}\times 16ds \pm 3pvv}$.

Caſe 1. When the Body aſcends, $\dot{t} = \frac{-8dq\dot{v}}{\overline{q-p}\times 16ds+3pvv}$, and (by Form 5.) the Fluent $t = \frac{-8dq\times{,}017453}{\sqrt{\overline{q-p}\times 48dsp}}$ × Degrees in the Arch whoſe Tangent is $\sqrt{\frac{3pvv}{\overline{q-p}\times 16ds}}$. But when $t = 0$, $v = b$; and the Fluent corrected is

$t =$

$t = \frac{2dq \times ,017453}{\sqrt{\overline{q-p} \times 3dsp}} \times$ Degrees in the Difference of the Arches whoſe Tangents are $\sqrt{\frac{3pbb}{\overline{q-p} \times 16ds}}$ and $\sqrt{\frac{3pvv}{\overline{q-p} \times 16ds}}$; therefore when $v=0$, the whole Time of Aſcent $t = \frac{2dq \times ,017453}{\sqrt{\overline{q-p} \times 3dsp}} \times$ Degrees in the Arch whoſe Tangent is $\frac{b}{4}\sqrt{\frac{3p}{\overline{q-p} \times ds}}$.

Likewiſe $\dot{x} = \frac{-8dqv\dot{v}}{\overline{q-p} \times 16ds + 3pvv}$, and (by Form the 4th) the Fluent $x = \frac{-2.3025 \times 8dq}{6p}$ Log: $\overline{q-p} \times 16ds + 3pv^2$: And when duly corrected $x = \frac{4dq \times 2.30258}{3p} \times$ Log: $\frac{\overline{q-p} \times 16ds + 3pbb}{\overline{q-p} \times 16ds + 3pvv}$.

Caſe 2. When the Globe deſcends $\dot{t} = \frac{8dq\dot{v}}{\overline{q-p} \times 16ds - 3pvv}$, and (by Form the 6th) $t = \frac{dq \times 2.302585}{\sqrt{\overline{q-p} \times 3dsp}} \times$ Log: $\frac{\sqrt{\overline{q-p} \times 16ds} + \sqrt{3pvv}}{\sqrt{\overline{q-p} \times 16ds} - \sqrt{3pvv}} - \frac{\sqrt{\overline{q-p}.16ds} + \sqrt{3pbb}}{\sqrt{\overline{q-p}.16ds} - \sqrt{3pbb}}$: and this laſt Term vaniſhes when b is o, or the Body deſcends from Reſt.

Alſo $\dot{x} = \frac{8dqv\dot{v}}{\overline{q-p} \times 16ds - 3pvv}$; whence the Fluent

FIG. Fluent $x = \frac{8dq\times 2.3025}{-6p}$ Log: $\overline{q-p}\times 16ds - 3pvv$: and when corrected, $x = \frac{4dq\times 2.302585}{3p}\times$ Log: $\frac{\overline{q-p}\times 16ds - 3pbb}{\overline{q-p}\times 16ds - 3pvv}$: where b is $=0$, when the Globe deſcends from Reſt.

COR. 1. *The greateſt Velocity the Globe can acquire by an infinite Deſcent is* $\sqrt{\frac{q-p}{3p}\times 16ds}$: For when x or t is infinite the Denominator $\overline{q-p}\times 16ds - 3pvv = 0$.

COR. 2. *Let* $G = q\sqrt{\frac{d}{\overline{q-p}\times 3ps}}$.

$$H = 4\sqrt{\frac{\overline{q-p}\times ds}{3p}}.$$

$N =$ *Number of the Log.* $\frac{.434294t}{G}$.

Then $t = 2.30258G\times$ *Log.* $\frac{H+v}{H-v}$ *or* $N = \frac{H+v}{H-v}$, *which reduced gives* $v = \frac{N-1}{N+1}H$, *when the Globe deſcends.*

COR. 3. $x = 2.302585\times\frac{4dq}{3p}\times$ *Log.* $\frac{HH}{HH-vv}$ $= 2.302585\times\frac{4dq}{3p}\times$ *Log.* $\frac{\overline{N+1}^2}{4N}$, *when the Globe deſcends.*

COR. 4. *In like Manner the Velocity and Space may be found from the Time when the Globe aſcends.*

SCHOLIUM.

If we had ſuppoſed the Reſiſtance to be as any Power of the Velocity v^n, then we ſhould find (as in Cor. 3. of the laſt Prob.) $\frac{3pv^nW}{16dsqb^{n-2}}$ = *Reſiſtance of the Globe with Velocity* v: and conſequently $\dot{t} = \frac{\mp 8dqb^{n-2}\dot{v}}{\overline{q-p} \times 16dsb^{n-2} \pm 3pv^n}$; and $\dot{x} = v\dot{t} = \frac{\mp 8dqb^{n-2}v\dot{v}}{\overline{q-p} \times 16dsb^{n-2} \pm 3pv^n}$. And the Fluents will give t and x.

Note, the Denſity q muſt always exceed p, otherwiſe the Globe will not gravitate; contrary to the Suppoſition.

PROB. XV.

To find the Velocity and Reſiſtance of a Globe oſcillating in a Cycloid, in a reſiſting Medium.

Let Ba be the Arch deſcribed in one entire Oſcillation, C the loweſt Point, and CZ half the whole Cycloidal Arch equal to the Length of the Pendulum: Let the Globe deſcend from B, and put $CZ = a$, $CB = b$, $BD = x$, the reſt as in the laſt Problem. 218.

Then we ſhall find $\frac{q-p}{q}W$ = comparative Weight of the Globe in the Medium, and $\frac{3pvv}{16dsq}W$ = Reſiſtance of the Globe moving with the Velocity v, in the Point D; as in the former Problems. Now it is known that CZ is to CD, as the Weight of the Globe $\frac{q-p}{q}W$ is to its accelerating Gravity at D, which therefore is $\frac{q-p}{q}W \times \frac{b-x}{a}$. Therefore

FIG. 218. the whole Force by which the Pendulum is urged in D is $\frac{q-p}{aq}bW - \frac{q-p}{aq}Wx - \frac{3pv^2}{16dsq}W = y$. And therefore we ſhall find (the ſame as in Prob. XIV.) $\dot{t} = \frac{\dot{v}}{2sy}W$, and $\dot{x} = (v\dot{t} =) \frac{v\dot{v}}{2sy}W$. Put $f = \frac{q-p}{aq} \times 2s$, $g = \frac{3p}{8dq}$, and $b = \frac{q-p}{aq} \times \frac{3p}{4dq}W$: and then $\dot{t} = \frac{\dot{v}}{bf-fx-gvv}$, and $\dot{x} = \frac{v\dot{v}}{bf-fx-gvv}$: And the Fluents will give t and x.

1. Since $\dot{x} = \frac{v\dot{v}}{bf-fx-gvv}$, finding the Fluent (by the Help of Form the 4th and Rule 8. Prop. X.) we have $x = \frac{2.3025}{-2g}$ Log: $\overline{bf-fx-gvv+\frac{f}{2g}}$; but when $x=0$, $v=0$; therefore the Fluent corrected is $x = \frac{2.302585}{2g} \times$ Log: $\frac{bf+\frac{f}{2g}}{bf+\frac{f}{2g}-fx-gvv}$:

Let n = Number belonging to the Logarithm $\frac{2gx}{2.302585}$, and then $n = \frac{2bg+1}{2bg+1-2gx-\frac{2gg}{f}vv}$, which reduced gives $vv = :\frac{n-1}{n}b + \frac{n-1}{2ng} - x:\times\frac{f}{g}$.

2. Let $z = \frac{3pvv}{16dsq}W$ the Reſiſtance in D, then $vv = \frac{16dsq}{3pW}z$, and $vv = \frac{8dsq\dot{z}}{3pW}$; expunge v and $\dot{v}$ out of the Value of $\dot{x}$, and we ſhall have $\dot{x} = \frac{\dot{z}}{bb-bx-2gz}$, and the Fluent $x = \frac{2.3025}{-2g}$ Log: $bb-$

$bx - 2gz + \frac{b}{2g}$: and when duly corrected $x =$ FIG. 218.

$\frac{2.302585}{2g} \times \text{Log}: \dfrac{b + \frac{1}{2g}}{b + \frac{1}{2g} - x - \frac{2g}{b}z}$. Let

$n =$ Number of the Logarithm $\frac{2gx}{2.30258}$; then

will $n = \dfrac{b + \frac{1}{2g}}{b + \frac{1}{2g} - x - \frac{2g}{b}z}$, which reduced gives

$z = \frac{n-1}{n}b + \frac{n-1}{2gn} - x : \times$ into $\frac{b}{2g}$.

Cor. 1. *In the loweſt Point C, $n =$ Number belonging to the Logarithm $\frac{3pb}{2.30258 \times 4dq}$. And there the Velocity $= \sqrt{: \frac{\overline{n-1} \times 4dq}{3p} - b :} \times$ into $\frac{f}{ng}$, and the Reſiſtance $= : \frac{\overline{n-1} \times 4dq}{3p} - b : \times$ into $\frac{q-p}{naq}W$.*

Cor. 2. *But the Velocity and Reſiſtance are the greateſt, when $\dot{z}$ or $\overline{bb - bx - 2gz} \times \dot{x} = 0$, and thence $z = \frac{b-x}{2g}b = \overline{b-x} \times \frac{q-p}{aq}W$.*

Cor. 3. *And therefore the Velocity and Reſiſtance are the greateſt when $x = \frac{2.30258 \times 4dq}{3p} \times \text{Log}: \frac{3pb}{4dq} + 1:$*

For then $z = \overline{b-x} \times \frac{b}{2g} = \frac{n-1}{n} \times \overline{b + \frac{1}{2g}} - x:$ $\times \frac{b}{2g}$; which reduced gives $n = 2bg + 1$, and Log. n or the Number $\frac{2gx}{2.30258} =$ Log: of $\overline{2bg+1}$.

SCHOLIUM.

If the oſcillating Body is not a Globe, then the Proportion of its Reſiſtance to that of an equal Globe whoſe Diameter is d, muſt either be calculated from Prob. XXII. Sect. II. or found by Experiments; let that be as 1 to m: And then we muſt take $\frac{g}{m}$ inſtead of g, or $\frac{p}{mdq}$ inſtead of $\frac{p}{dq}$ in the foregoing Calculations.

PROB. XVI.

To find the Denſity of the Atmoſphere at any Height; ſuppoſing the Force of Gravity to be as any Power of the Diſtance from the Earth's Center, and the Denſity of the Air as the Compreſſion.

Let r = Radius of the Earth.
x = Any Diſtance from the Center.
d = Denſity of the Atmoſphere at the Earth's Surface.
z = Atmoſphere's Denſity at the Diſtance x.
n = Exponent of the Law of Gravity.

Since the Denſity is as the Preſſure therefore the Moment of the Denſity ∝ Moment of Preſſure, that is ∝ Moment of Matter × Force of Gravity: But Moment of Matter ∝ Denſity × Moment of Space. Therefore the Moment of Denſity ∝ Denſity and Moment of Space and Force of Gravity; that is $\acute{z} \propto z x^n \acute{x}$ univerſally.

Now it is collected from Experiments, that the Weight of 1 Foot high of Air at the Earth's Surface is to the Weight or Preſſure of the Atmoſphere, as FIG. 1 to $29725 = p$, at a mean Denſity; therefore, let $\acute{x}$ be any ſmall Height, and it is $p : d :: \acute{x} : \frac{d\acute{x}}{p} =$ Moment of Denſity at the Earth's Surface. Whence from the foregoing general Proportion, $\frac{d\acute{x}}{p} : dr^n\acute{x} :: -\acute{z} : zx^n\acute{x}$: therefore $\acute{z} = \frac{-x^n z\acute{x}}{pr^n}$, and $\frac{\dot{z}}{z} = \frac{-x^n\dot{x}}{pr^n}$: And (by Form the 1ſt and 2d) the Fluent is $2.3025 \text{ Log} : z : = \frac{-x^{n+1}}{\overline{n+1} \times pr^n}$. And duly corrected, $2.302585 \times \text{Log} : \frac{z}{d} : = \frac{r^{n+1} - x^{n+1}}{\overline{n+1} \times pr^n}$; therefore $z = d \times$ Number of the Log: $\frac{r^{n+1} - x^{n+1}}{2.302585 \times \overline{n+1} \times pr^n}$ $= d \times$ Number belonging to the Log. $\frac{r^{n+1} - x^{n+1}}{68444 \times \overline{n+1} \times r^n}$.

COR. 1. *If e be any ſmall Height above the Earth's Surface, then* $z = d \times$ *Number of the Log.* $\frac{-e}{68444}$.

COR. 2. *If* $n = 1$, *then* $z = d \times$ *Number of the Logarithm* $\frac{rr - xx}{68444 \times 2r}$.

COR. 3. *If* $n = 0$, $z = d \times$ *Number of the Logarithm* $\frac{r - x}{68444}$.

COR. 4. *If* $n = -2$, $z = d \times$ *Number belonging to the Logarithm* $\frac{r \times \overline{r - x}}{68444x}$. *In all which e, r and x are ſuppoſed to be taken in Feet.*

PROB.

PROB. XVII.

To find the Denſity of the Atmoſphere at any Height; ſuppoſing the Force of Gravity to be as any Power of the Diſtance, and the Compreſſion as any Power of the Denſity.

Let r = Radius of the Earth.
d = Denſity at the Earth's Surface.
x = any Diſtance.
z = Denſity at the Diſtance x, from the Center.
p = a Length of 29725 Feet.
n = Index of the Force.
m = Index of the Denſity.
v = compreſſing Force at the Diſtance x.

Now by the Hypotheſis $v \propto z^m$, and Force $\propto x^n$; and $mz^{m-1}\overset{\prime}{z} \propto$ Moment of Preſſure, that is as the Moment of Space and Denſity and Force of Gravity: that is $z^{m-1}\overset{\prime}{z} \propto x^n z\overset{\prime}{x}$, or $z^{m-2}\overset{\prime}{z} \propto x^n\overset{\prime}{x}$, univerſally.

To find the Moment of Denſity at the Earth's Surface, we have $\dot{v} \propto mz^{m-1}\dot{z}$; therefore (by Prop. II.) $v : \dot{v} :: z^m : mz^{m-1}\dot{z} :: z : m\dot{z}$, therefore $\dot{z} = \frac{z\dot{v}}{mv}$; but at the Earth's Surface, $z = d$; and (taking any very ſmall Space $\overset{\prime}{s}$) it will be, $p : v :: \overset{\prime}{s} : \overset{\prime}{v} :: \dot{s} : \dot{v}$, and $\frac{\dot{v}}{v} = \frac{\dot{s}}{p}$; whence $\dot{z} = \frac{d\dot{s}}{mp}$ = Fluxion of Denſity at the Earth's Surface: Therefore from the univerſal Proportion; $d^{m-2} \times \frac{ds'}{mp} : r^n\overset{\prime}{s} :: z^{m-2}\overset{\prime}{z}$:

—

$-x^n\dot{x} :: z^{m-2}\dot{z} : -x^n\dot{x}$: Whence $z^{m-2}\dot{z} = \frac{-d^{m-1}x^n\dot{x}}{mpr^n}$; and the Fluent (by Form the 2d) is $\frac{z^{m-1}}{m-1} = \frac{-d^{m-1}x^{n+1}}{\overline{n+1} \times mpr^n}$: And the Fluent corrected is $\frac{z^{m-1} - d^{m-1}}{m-1} = \frac{d^{m-1}}{\overline{n+1} \times mpr^n} \times \overline{r^{n+1} - x^{n+1}}$:

And by Reduction $z^{m-1} = d^{m-1} + \frac{\overline{m-1} \times r}{\overline{n+1} \times mp}d^{m-1}$

$- \frac{\overline{m-1} \times x^{n+1}}{\overline{n+1} \times mpr^n}d^{m-1}$.

PROB. XVIII.

To find the Diameters of the Earth.

Suppoſe the Earth to be in the Form of the Spheroid, *APBQ*; *AB* the Equinoctial, *PQ* the Axis; and let its mean Radius $CR = 1$, $AC = 1 + v = a$. And $PC = 1 - v = e$, let *CS* be the Conjugate to *RC*, and *RT* perpendicular to *CS*, then by Conics $CS = \sqrt{aa + ee - 1} = 1 + vv = c$, and $RT = \frac{ae}{c} = 1 - 2vv = p$. Here I reject the Powers of v above vv as being very inconſiderable. 227.

In the mean Place *R* a heavy Body falls about 16,0917 Feet in 1 Second; and the verſed Sine of the Arch deſcribed by *R* in 1 Second by the Earth's Revolution is ,04, (if $RC = 21000000$ Feet); alſo as $1 : \sqrt{\frac{1}{2}} :: ,04 : ,0283$ = to the centrifugal Force in *R*, as 16,0917 repreſents the Force of Gravity; therefore

FIG. 227. therefore $16{,}12 = d$, will be the gravitating Force at R if the Earth ſtands ſtill: And this is nearly the ſame with that of a Spheroid whoſe Axis is $2RT$; and Radius of the (Baſe or) greateſt Circle $\sqrt{ac}$; which (by Corol. 2. Prob. XI.) is $2p - \frac{4pp}{3\sqrt{ac}} - \frac{2pp}{5ac\sqrt{ac}} \times \overline{ac - pp} = \frac{2}{3} + \frac{4}{15}v - \frac{187}{30}v^2$, omitting the given Quantities in that Corollary. Alſo the Force of the Earth at A is nearly the ſame as a Spheroid whoſe Axis is AB and Radius of the Baſe $\sqrt{ae}$, that is (by the ſame Cor.) $2a - \frac{4aa}{3\sqrt{ae}} + \frac{2aa \times \overline{aa - ae}}{5ae\sqrt{ae}} = \frac{2}{3} + \frac{2}{15}v + \frac{2}{5}vv$. Likewiſe the Force of the Earth at P is $2e - \frac{4ee}{3a} - \frac{2ee \times \overline{aa - ee}}{5a^3} = \frac{2}{3} + \frac{2}{5}v + 4vv$.

For the centrifugal Force at the Equinoctial, it is as $\sqrt{\frac{1}{2}} : 1 + v :: {,}04 : {,}05657 \times \overline{1 + v} = b + bv$ (by Subſtitution) $=$ centrifugal Force at A: Alſo $\frac{2}{3} + \frac{4}{15}v - \frac{187}{30}vv : d :: \frac{2}{3} + \frac{2}{15}v + \frac{2vv}{5} : \overline{1 - \frac{1}{5}v + 10{,}03vv} \times d$ for the gravitating Force at A, if the Earth ſtood ſtill: from this take $b + bv$ the centrifugal Force, and we get $\overline{1 - \frac{1}{5}v + 10{,}03vv} \times d - b - bv$ for the Force of Gravity at A when the Earth is in Motion.

Let $CD = x$, $CE = y$. Since the Gravity and alſo the centrifugal Force (which is as the Decreaſe of Gravity) in A and D, are as a or $1 + v$ to x, therefore the gravitating Force of the Earth in D, when the Earth is in Motion will be $\overline{1 - \frac{6}{5}v + 11.23v^3} \times dx - bx$.

Laſtly, $\frac{2}{3} + \frac{4}{15}v - \frac{187}{30}vv : d :: \frac{2}{3} + \frac{2}{5}v + 4vv : \overline{1 + \frac{7}{5}v + 15.27vv} \times d$ the Force of Gravity at P. And ſince the Forces in P and E are as e or $1 - v$ to y, therefore the Force at E is $\overline{1 + \frac{6}{5}v + 16{,}47vv} \times dy$.

Now

FIG. 227.

Now ſuppoſe the Weights of the Columns x and y to be equal; therefore their Moments or Fluxions multiplied into the gravitating Forces at D and E, will be equal; that is $\overline{1-\frac{6}{5}v+11.23vv}\times dx\dot{x}-bx\dot{x}=\overline{1+\frac{6}{5}v+16.47vv}\times dy\dot{y}$, and taking the Fluents, and dividing by $\frac{1}{2}d$, and putting $1+v$ and $\overline{1-v}$ for x and y, there comes out $\overline{1-\frac{6}{5}v+11.23v^2-\frac{b}{d}}\times\overline{1+v}^2=\overline{1+\frac{6}{5}v+16.47vv}\times\overline{1-v}^2$; that is $\overline{1-\frac{b}{d}}+\overline{\frac{4}{5}-\frac{2b}{d}}\times v+\overline{9.83-\frac{b}{d}}\times v^2=1-\frac{4}{5}v-15.07v^2$, Or $vv+{,}0639v={,}000141$, whence $v={,}00213$, and $CA-CP={,}00426$: And therefore if the mean Radius of the Earth be 21000000, then $CA-CP$ $=89460$ Feet or 17 Engliſh Miles nearly: Therefore $AC=21044730$, and $PC=20955270$ Feet.

SCHOLIUM.

This Computation ſuppoſes the Earth every where of equal Denſity: But ſince that is not certainly known, nor with what Force a Spheroid accurately attracts a Body when ſituated out of the Axis; nor whether the Earth itſelf is exactly a Spheroid: Theſe Things may render this Solution a little incorrect. If the Earth be more rare towards the Equinoctial than towards the Poles; its Height at the Equinoctial will be increaſed in that Proportion.

PROB. XIX.

FIG. 247. *The Velocity and Direction of a Projectile being given*; *to find the Path it deſcribes in a reſiſting Medium.*

Let *AC* be the Direction of the Projectile, *ABF* the Curve deſcribed; draw *AD* parallel, and *CBD* perpendicular to the Horizon, and *cO* infinitely near *CB*. Alſo draw *Bm* parallel to *Cc*, and *Bn* parallel to *AD*.

Call *AC*, x; *CB*, y; *AB*, z; and let m and n be the Sine and Coſine of *DAC*; $b =$ Velocity at *A*, or the Space deſcribed in Time 1; $h =$ Space deſcended thro' in the Time 1, by the Gravity 1, acquiring the Velocity $2h$, in Vacuo; $c =$ Velocity when the Reſiſtance is equal to the Weight in the Fluid, (ſee Cor. 1. Prob. 14.); $V =$ Velocity at *C* in the Line *AC*; $v =$ the Velocity at *B* in the Curve $= \frac{\dot{z}V}{\dot{x}}$, by Reſolution of Motion.

Since the Reſiſtance is as the Square of the Velocity, therefore cc (Vel.) : 1 (Reſ.) : : $vv : \frac{vv}{cc} =$ Reſiſtance in Direction *BO*, and $\dot{z} : \dot{x} : : \frac{vv}{cc}$ or $\frac{V^2\dot{z}^2}{cc\dot{x}^2} : \frac{V^2\dot{z}}{cc\dot{x}} =$ Reſiſtance in Direction *Bm*, by the Reſolution of Forces.

The Body is drawn from the Tangent at *B* by its Gravity alone; and $\frac{1}{2}\ddot{y}$ is the Space it deſcends thro', in the Time of deſcribing $\dot{x}$ with Velocity V; and twice that Space, or $\ddot{y}$, would be deſcribed uniformly in the ſame Time; and the Velocities being as the Spaces uniformly deſcribed; we have, $\dot{x} : V : : \ddot{y} :$

$V\ddot{y}$

FIG. 247.

$\frac{V\ddot{y}}{\dot{x}}$ = Velocity generated by Gravity in that Time.

Now ſince Gravity does not affect the Motion along *AC*, the Retardation therein is wholly owing to the Reſiſtance; therefore $-\dot{V}$ being the Velocity deſtroyed by the Reſiſtance alone; and $\frac{V\ddot{y}}{\dot{x}}$ the Velocity generated by the Gravity alone; and the Velocities generated or deſtroyed being as the Forces, it will be, 1 (Gravity) : $\frac{V\ddot{y}}{\dot{x}}$ (Vel.) : : $\frac{V^2\dot{z}}{cc\dot{x}}$ (Reſ.) : $-\dot{V}$ (Vel.); whence $\frac{-\dot{V}}{V^3} = \frac{\dot{z}\ddot{y}}{cc\dot{x}^2}$.

Put $\dot{y} = s\dot{x}$, ſuppoſing $\dot{x}$ given, then $\ddot{y} = \dot{s}\dot{x}$, therefore $\frac{-\dot{V}}{V^3} = \frac{\dot{z}\dot{s}}{cc\dot{x}}$. But $\dot{z}^2 = \dot{x}^2 + \dot{y}^2 - 2m\dot{x}\dot{y} = \dot{x}^2 + s^2\dot{x}^2 - 2ms\dot{x}^2$, whence $\frac{-\dot{V}}{V^3} = \frac{\dot{s}}{cc}\sqrt{1-2ms+ss}$; and (by Form 27.) the Fluent is $\frac{1}{2VV} = \frac{1-mm}{2cc}L \times \text{Log: } s-m+\sqrt{1-2ms+ss} : + \frac{s-m}{2cc}\sqrt{1-2ms+ss}$; and corrected, $\frac{1}{VV} - \frac{1}{bb} = \frac{nn}{cc}L \times \text{Log: } \frac{s-m+\sqrt{1-2ms+ss}}{1-m} : + \frac{s-m}{cc}\sqrt{1-2ms+ss} + \frac{m}{cc}$; for in *A*, $s=0$.

Again, Velocity × Flux. Velocity ∝ Force × Flux. Space, (ſee Prob. I.) ;and if $2b$ repreſents the Velocity generated by Gravity, in deſcribing any Space S; then $\dot{s}$ will repreſent the Fluxion of the Velocity; therefore, $2b \times \dot{s}$ (Vel. × Fl. Vel.) : $1 \times \dot{s}$ (Grav. × Space) : : $-V\dot{V} : \frac{VV\dot{z}}{cc\dot{x}} \times \dot{x}$, whence $\dot{z} = \frac{-cc\dot{V}}{2bV}$; and the correct Fluent, $z = \frac{ccL}{2b} \times \text{Log: } \frac{b}{V}$.

FIG. 247. Now ſince $\frac{-\dot{V}}{V} = \frac{2b\dot{z}}{cc}$, and $\frac{-\ddot{V}}{V^3} = \frac{\dot{z}\ddot{y}}{cc\dot{x}\dot{x}}$, if the latter be divided by the former, we have $\frac{1}{VV} = \frac{\ddot{y}}{2b\dot{x}\dot{x}}$, or $2b\dot{x}^2 = VV\ddot{y}$, in Fluxions, $2V\dot{V}\ddot{y} + VV\dddot{y} = 0$, or $\frac{-\dot{V}}{V} = \frac{\dddot{y}}{2\ddot{y}} = \frac{2b\dot{z}}{cc}$, or $cc\dddot{y} = 4b\ddot{y}\dot{z}$, an Equation between *y* and *z*. Or $cc\dddot{y} = 4b\ddot{y}\sqrt{\dot{x}^2+\dot{y}^2-2m\dot{x}\dot{y}}$, an Equation between *x* and *y*.

Now ſince $s = \frac{\dot{y}}{\dot{x}} = \frac{S.OBm}{S.BOm}$, and ſince the Angle at *m* is given; therefore if we aſſume any Angle for *B*, then ∠ *O* will be given, and conſequently *s*, and from thence *V*; and from *V*, *z* is known.

Let $E = \frac{nnL}{cc} \times \text{Log:}\ \frac{s-m+\sqrt{1-2ms+ss}}{1-m}: + \frac{s-m}{cc}\sqrt{1-2ms+ss} + \frac{m}{cc}$; then $\frac{1}{VV} = \frac{1}{bb} + E$, and $VV = \frac{bb}{1+bbE}$. Alſo $z = \frac{ccL}{2b} \times \text{Log:}\ \frac{b}{V} = \frac{ccL}{4b} \times \text{Log:}\ \frac{bb}{VV} = \frac{ccL}{4b} \times \text{Log:}\ 1+bbE$: and $\dot{z} = \frac{bbcc\dot{E}}{4b \times \overline{1+bbE}} = \frac{bb\dot{s}\sqrt{1-2ms+ss}}{2b \times \overline{1+bbE}}$, and $\dot{x} = \frac{\dot{z}}{\sqrt{1-2ms+ss}} = \frac{bb\dot{s}}{2b \times \overline{1+bbE}}$. Alſo $\dot{y} = s\dot{x} = \frac{bbs\dot{s}}{2b \times \overline{1+bbE}}$. Likewiſe $v = \frac{\dot{z}}{\dot{x}}V = V\sqrt{1-2ms+ss} = b\sqrt{\frac{1-2ms+ss}{1+bbE}}$.

Hence if you take $s = r + u$, and $\dot{s} = \dot{u}$, and if r be taken ſucceſſively = any given Quantities in Arithmetic Progreſſion, and the Value of E be found in a Series, and ſo proceed by Rule 5, Prob. X. you will get all the Parts of x and y. FIG. 247.

Cor. 1. *In the Vertex F,* $s = m$, *and* $E = \frac{nnL}{cc} \times Log : \sqrt{\frac{1+m}{1-m}} : + \frac{m}{cc}$. *Therefore put* $T = Log :$ *Cotan. of half the* $\angle ACD - 10$, *as found in the Tables; and then E or e* $= \frac{nnLT+m}{cc}$; *and* $V = \frac{b}{\sqrt{1+bbe}}$, $v = \frac{bn}{\sqrt{1+bbe}}$, *and* $z = \frac{ccL}{2b} \times Log : \sqrt{1+bbe} :$

For by Trigonometry, if $T =$ Cotan. $\frac{1}{2}C$, the Coſine of its double, (that is the Sine of A) will be $\frac{TT-1}{TT+1} = m$, reduced $T = \sqrt{\frac{1+m}{1-m}} =$ Cotan. $\frac{1}{2}C$.

Cor. 2. *If the Projectile be a Cannon Ball, and if* δ *be the Diameter, and* $\varkappa$ *the Denſity of another ſmall light Ball, which projected at the ſame Elevation, with the Velocity* $\beta = b\sqrt{\frac{\varkappa - p \times \delta}{q - p \times d}}$, *(ſee Prob. XIV.) This Ball will deſcribe a Curve in the Air ſimilar to the other Curve deſcribed by the Cannon Ball; and whoſe Height or Baſe will be to that of the other reſpectively; as* dq *to* $\delta\varkappa$.

To prove this, let lt be the *Element* of the Curve, lrs the Tangent at l, rt perpendicular to the Horizon. Since this Figure is ſimilar to the correſpondent *Element* of the other Curve, therefore the Lines lr, rs, rt, are in given Ratio's to one another. Now the Line rt ($\ddot{y}$) is generated by the Gravity, and sr ($\ddot{z}$) deſtroyed by the Reſiſtance, both in the Time ($\dot{t}$) wherein lr ($\dot{z}$) is deſcribed with Velocity v. Therefore $sr \propto \frac{\text{Reſiſtance}}{\text{Matter}} \propto \frac{vvdd}{d^3q} \propto \frac{vv}{dq}$. Alſo $rt \propto \frac{\text{Gravity}}{\text{Matter}}$

$\propto$

FIG. 247. $\propto \frac{q-p}{q} \times \frac{W}{W} \propto \frac{q-p}{q}$. And *sr* being in a given Ratio to *rt*, $\frac{vv}{dq} \propto \frac{q-p}{q}$, or $vv \propto \overline{q-p} \times d$, therefore $vv : VV :: \overline{q-p} \times d : \overline{\kappa-p} \times \delta :: bb : \beta\beta$, (*V* being the Velocity in the other Curve); therefore $\beta = b \sqrt{\frac{\overline{\kappa-p} \times \delta}{\overline{q-p} \times d}}$.

Again, Time (in *lr* or *rt*) $\propto \sqrt{\frac{rt}{\text{Force of Gravity}}} \propto \sqrt{\frac{lr \times q}{q-p}}$, and $lr \propto v \times \text{Time} \propto \sqrt{\frac{lr \times q}{q-p} \times \overline{q-p} \times d} \propto \sqrt{lr \times dq}$; whence $lr^2 \propto lr \times dq$, or $lr \propto dq$. But if the *Elementa* of two Curves be ſimilar, the whole Curves will be ſo, whence $lr \propto$ whole Baſe of the Curve; therefore the Baſe is as dq; whence, 1 Baſe : 2 Baſe :: $dq : \delta\kappa$.

COR. 3. *If the two Balls projected with the Velocities* b *and* β, *were to move without any Reſiſtance in the Medium; they would deſcribe two ſimilar Parabolas, whoſe Baſes or Heights are reſpectively as thoſe of the other two Curves deſcribed in the reſiſting Medium; and* β *may be taken* $= b \sqrt{\frac{\kappa-p}{qd}}\,\delta$, *as* q *is vaſtly greater than* p; *or even* $b\sqrt{\frac{\kappa\delta}{qd}}$, *if* κ *very much exceeds* p. *And if the Gravity of the Balls was not at all diminiſhed in the Fluid; the Velocities would be exactly as* $\sqrt{dq}$ *and* $\sqrt{\delta\kappa}$.

That they would deſcribe Parabolas in a non-reſiſting Medium, is plain from the Theory of Projectiles; which will be ſimilar, becauſe projected at the ſame Angle of Elevation. But the horizontal Diſtance of ſuch a Projection, is as the Velocity Square directly, and the Gravity reciprocally, which in the two Balls will be as $bb \times \frac{q}{q-p}$ to $\beta\beta \times \frac{\kappa}{\kappa-p}$ or $bb \times$

$\kappa - p$

FIG. 247.

$\frac{\overline{\varkappa-p}\times\delta}{\overline{q-p}\times d}\times\frac{\varkappa}{\varkappa-p}$, that is as dq to $\delta\varkappa$, which was proved to be the Ratio of the Baſes of the other two Curves. Alſo it was proved before, that $\frac{vv}{dq} \propto \frac{q-p}{q}$, but $\frac{q-p}{q}$ repreſents the relative Gravity in the Fluid; which if it was $=1$, then would $vv \propto dq$.

Cor. 4. *And hence we have a Method of determining the Curve deſcribed by a Cannon Ball in the Air, from the Phænomena.* For if a Cannon Ball whoſe Diameter is d and ſpecific Gravity q, is projected with Velocity b. Make a ſmall Globe of Cork, or rather of Paper, Bladder, *&c.* made hollow, and whoſe Diameter is δ, and ſpecifick Gravity $\varkappa$, and colour it red or black; if this be projected in the Air, with the Velocity β or $b\sqrt{\frac{\varkappa-p}{qd}\delta}$, and at the ſame Elevation; it will exhibit to View, the very Curve deſcribed by the Cannon Ball, in Miniature. And if a Leaden Ball be projected along with it, you will have all the Phænomena of the Cannon Ball in the Air. And obſerving where the two Globes fall, if the Diſtances of the horizontal Projections be meaſured, you will have the Space that the Cannon Ball falls ſhort, by Reaſon of the Reſiſtance of the Medium. But the Leaden Ball ought rather to be projected with the Velocity $\beta\sqrt{\frac{\varkappa}{\varkappa-p}}$ or $b\sqrt{\frac{\varkappa\delta}{qd}}$; becauſe the Metal Ball looſes no ſenſible Weight in the Air, and it ought to be projected as far with the Gravity 1, as the Cork Ball is with the Gravity $\frac{\varkappa}{\varkappa-p}$, without Reſiſtance. But if a very ſlender Body can be made, like an Arrow, ſomething heavier before, and of the ſame ſpecifick Gravity in the whole, as the Cork Ball; then this

and

FIG. 247. and the Cork Ball may be projected together, with the ſame Velocity; and this ſlender Body will be very little reſiſted by the Air, as well as the Leaden one, and will ſhow how far the Ball would be projected without Reſiſtance. And the ſame all holds good if the Balls be projected perpendicularly upwards with the ſame Velocity; then it will appear how far the Cork Ball comes ſhort of the other. And inſtead of dq and δx, you may take the Weights divided by dd and $\delta\delta$: for $dq \propto \frac{w}{dd}$, w being the Weight in Vacuo.

SCHOLIUM.

This Problem ſuppoſes the Air every where of equal Denſity; but in a Ball ſhot from one of our greateſt Guns, to the Diſtance of four or five Miles, the Height at the greateſt horizontal Diſtance, will be about a Mile, and there the Denſity of the Air is about a ſixth Part leſs than that at the Surface; ſo that the Ball is leſs reſiſted there, and conſequently flies further. Likewiſe the Denſity of the Air is different at different Times, and conſequently the Reſiſtance will be ſomething different upon that Account.

PROB.

PROB. XX.

If Water or any Fluid aſcends and deſcends alternately with a reciprocal Motion in the Legs KL, MN of a cylindrical Canal or Pipe; to find the Time of one Libration or Reciprocation of the Water. FIG. 248.

Let *AD* be the Level or horizontal Line in which the Surface of the Water ſtands when at reſt; and let the Water deſcend from *F* in the Leg *KL*, and aſcend from *C* in the other Leg; and ſuppoſe it to be in *E* and *G* at the ſame Time. Draw *EB*, *DH* perpendicular to the horizontal Lines *BA*, *GH*, *and put AF or CD* $=b$, *AE or GD* $=x$, *Sine of the Angles L, N* $=p$, q; $v =$ *Velocity in E*, $t =$ *Time of deſcending thro' FE*; $l =$ *Length of the Canal A L N D*, $w =$ *its Weight*, $s = 16\frac{1}{12}$ *Feet:* The Force with which the Motion of the Water is accelerated and retarded alternately, is the perpendicular Height of the Water at *E* above *G*. Then, becauſe $EB = px$, and $DH = qx$, the Preſſure of the two Columns *EA*, *DG* is $\frac{p+q}{l}wx$, the accelerating Force. By Prob. I. $v\dot{v} \propto \frac{F\dot{x}}{b}$, and in falling Bodies, when $v = 2s$, $\dot{v} = \dot{S}$, whence $2s \times \dot{S} : \frac{w\dot{S}}{w} :: v\dot{v} : \frac{p+q}{l}wx \times \frac{-\dot{x}}{w}$, whence $v\dot{v} = \frac{-\overline{p+q}}{l} \times 2sx\dot{x}$, and $\frac{v^2}{2} = -\frac{p+q}{2l} \times 2sxx$, but in *F*. $v = 0$, and $x = b$, therefore the Fluent corrected is $vv = \frac{p+q}{l} \times 2s \times \overline{bb - xx}$. Alſo (Prob. I.) $\dot{t} \propto \frac{\dot{S}}{v}$ and

FIG. 248. and in falling Bodies $\dot{t} = \frac{\dot{s}}{2s}$, whence $\frac{\dot{s}}{2s} : \frac{\dot{s}}{2s} :: \dot{t} : \frac{\dot{x}}{v}$, therefore $\dot{t} = \frac{\dot{x}}{\sqrt{\frac{p+q}{l} \times 2s}\ \sqrt{bb-xx}} = \sqrt{\frac{l}{\overline{p+q} \times 2s}} \times \frac{\dot{x}}{\sqrt{bb-xx}}$. And by Form 10, $t = \sqrt{\frac{l}{\overline{p+q} \times 2s}} \times$ Arch, whoſe Radius is 1, and Sine $\frac{x}{b}$, and when $x = b$, that Arch $= \frac{3.1416}{2}$; whence $t = \sqrt{\frac{l}{\overline{p+q} \times 2s}} : \times \frac{3.1416}{2}$, the Time of deſcending from *F* to *A*, and $2t = 3.1416\sqrt{\frac{l}{\overline{p+q} \times 2s}}$, the Time of one Libration of the Water.

COR. 1. *Comparing this Value of t with that in Prob. VII. and Cor. 2. it appears, that the Time of one Libration or Undulation of the Water, is equal to the Time of Oſcillation of a Pendulum, whoſe Length is* $\frac{l}{p+q}$; *which Length is* $= \frac{1}{2}l$, *when L and N are right Angles.*

COR. 2. *All the Times of Libration are equal, whether the Height aſcended and deſcended be greater or leſs.*

COR. 3. *It matters not what Figure the Pipe is of below the Level of C, provided the Parts above C be ſtreight wherein the Surface of the Water moves.*

COR. 4. *The Water in one Libration will deſcend to I, ſo that AI=AF. For when* $v = 0$, $bb - xx = 0$, *and* $x = +b$ *or* $-b$.

SCHOLIUM.

If the Leg *MN* be wider than *KL*, and its Section to that of *KL* as *c* to 1, we ſhall get $v\dot{v} =$

p+

FIG. 248.

$\frac{p+\frac{q}{c}}{l+x-\frac{x}{c}} \times -2sx\dot{x}$. And if c be infinite, (or which is the ſame) the End *M* immers'd in ſtagnant Water, then $v\dot{v} = \frac{-2psx\dot{x}}{l+x}$.

There is a great Analogy between the Libration of the Water in the Canal *KLMN*, and the Waves of the Sea, *&c.* Therefore if a Pendulum be made whoſe Length is the Breadth of a Wave from Top to Top; then in the Time that it performs one Oſcillation, the Waves will advance forward a Space equal to their Breadth.

For (by Cor. 1.) in the Time that the Pendulum $(\frac{1}{2}l)$ vibrates once, the Water in the Canal will make one Libration; and (the Length of the Canal being ſuppoſed $= \frac{1}{2}$ the Breadth of a Wave,) in the ſame Time, the Water in the Hollow, or loweſt Place between two Waves, will become the higheſt, that is, the Wave will move forward half its Breadth. And in two Vibrations (or one Vibration of the Pendulum $2l$) will advance a Space equal to the whole Breadth.

PROB. XXI.

FIG. 249. *Let a Bar of Steel (perfectly elaſtic) be ſupported at both Ends; and allowing that* 10 *Hundred Weight Avoirdupoiſe hanging in the Middle, will juſt break the ſaid Bar; it is required to find the Weight of a Globe, falling perpendicular* 185 *Feet on the Middle of the Bar, to have the ſame Effect.*

Let *DBE* be the Bar of Steel, ſupported at the Ends *D*, *E*. Now before this Prob. can be ſolved, ſomething more muſt be aſſumed than is here expreſſed. Suppoſe then, that the Space *BA* is known, thro' which the Bar is bent before it breaks; and let the Bar, from the Poſition *DBE*, be put into the Poſition *DAE*, or very near it, and then the 10 C. Weight laid ſoftly upon *A*, which juſt breaks it. Now the Weight falling upon it in the Direction *FB*, is to bend the Bar juſt thro' the Space *BA*, to *A*, where it breaks.

Let $c =$ *the Weight* (10 C.) *capable to break it when ſuſpended at A.*

$w =$ *Weight ſought, which is to break it by falling.*

$a = 16\frac{1}{12}$ *Feet.*

$d = FB$ *the Height that* w *falls* $= 185$ *Feet.*

$b = BA$, *the Space the Bar is bent through when it breaks.*

then $2\sqrt{ad} =$ *Velocity of* w *at the Point B, (the Velocity being meaſured by the Feet deſcribed in a Second.)*

$BC = x$, $Cc = \dot{x}$.

Now by the Principles of Mechanics, when the Bar is bent into the Poſition *DCE*, it exerts a Force

which is as the Diſtance BC. Therefore $BA\,(b)$: FIG. 249.
Force at $A\,(c)$: : $BC\,(x)$: $\frac{cx}{b}$ = Force at C. Therefore the Force that w is acted on at C is $\frac{cx}{b} - w$. Alſo by Mechanics, $v\dot{v} \propto \frac{F\times\dot{s}}{B}$, ſee Prob. I. In the Caſe of falling Bodies, if $2a$ is the Velocity gained in a Second, $\dot{x}$ is the Velocity gained in moving over $\dot{x}$ at the End of 1 Second. And the Weights of Bodies being as the Quantities of Matter, therefore (from the general Proportion) $2a\dot{x}$ (Vel.×Fl.Vel.) : $\frac{w\dot{x}}{w}$ $\left(\frac{\text{Weight}\times\text{Fl. Space}}{\text{Body}}\right)$: : $-v\dot{v}$: $\frac{\frac{cx}{b}-w}{w}\times\dot{x}$, hence $-v\dot{v} = \frac{cx-bw}{bw}\times 2a\dot{x}$, and the Fluent is $-\frac{vv}{2} = \frac{\frac{1}{2}cx^2-bwx}{bw}\times 2a$, but at B, $v = 2\sqrt{ad}$, and $x=0$, therefore the Fluent corrected is $4ad-vv = \frac{cx^2-2bwx}{bw}\times 2a$; but in A, $v=0$, and $x=b$, therefore $4ad = \frac{cbb-2wbb}{bw}\times 2a$, or $2d = \frac{c-2w}{w}b$, and $2dw = bc-2bw$, whence $w = \frac{bc}{2d+2b}$, or $w = \frac{bc}{2d}$ nearly.

COR. 1. *Half the Weight that will break the Bar at A, will break it when laid upon B. For when d* $=0$, $w = \frac{bc}{2b} = \frac{1}{2}c$.

COR. 2. *The Weight, that by falling a given Height will break any Bar, is very nearly as the Space* (b) *thro' which the Bar will bend before it breaks. Hence the Reaſon why brittle Bodies break ſooner by Percuſſion, than others of equal Strength:*

FIG. 249. COR. 3. *Hence the Weight w is ceteris paribus, reciprocally as d the Height fallen from, very near.*

SCHOLIUM.

Here no Notice is taken of the Weight of the Bar itſelf. If inſtead of deſcending, the Weight w was ſuppoſed to be projected horizontally, with a certain Velocity againſt the Bar *DBE*, then the Force at *C* would be $\frac{cx}{b}$, and $-v\dot{v} = \frac{cx}{b} \times \frac{\dot{x}}{w} \times 2a$, and $4ad - vv = \frac{cx^2}{bw} \times 2a$. And when $v = 0$, $x = b$, and thence $4ad = \frac{cb}{w} \times 2a$, and $w = \frac{bc}{2d}$, exactly, where Gravity has no Concern.

PROB. XXII.

250. *If any Number of Ivory Balls of an Inch Diameter be ſuſpended in a right Line, and the firſt A let fall from any Height upon the next; to find the Velocity of the Motion propagated thro' them.*

Here we ſuppoſe that the Balls are perfectly elaſtic, and placed ſo as juſt to touch each other, or very near it. But before this Problem can be ſolved, ſuch an Experiment as this muſt be made; black the firſt Ball over with Ink, and let it fall upon the ſecond in an Arch of a Circle from the Height deſign'd, or with the given Velocity; this will make a black Spot on the other; take its Diameter, and from thence find the Altitude of the Segment, or the verſed Sine of half that Spot. Now the Force by which this Motion is communicated from one Ball to

FIG. 250.

to another, is the Elaſticity of the ſmall Segment which is flatten'd or dented in by the Stroke, and acts againſt the other Ball by Way of a Spring, in like Manner as is deſcribed in Ex. 8. Prob. XIII. Sect. I. and this Force is as the Height of the Segment, drove in by the Stroke. Now in two Balls meeting each other, there are two Segments ſtruck in, both which exert their joint Force in ſeparating the Balls again, after meeting.

Let w = *Weight of a Ball.*
p = *the Weight which can make an equal Compreſſion in the Globe, as the Stroke did.*
a = *BC, which repreſents the verſed Sine of the Segment ſtruck in, C being the Vertex.*
d = *Diameter of a Globe or Ball as A.*
c = *Velocity it is projected with at firſt.*
s = $16\frac{1}{12}$ *Feet.*
x = *CD.*
v = *Velocity in D.*
t = *Time of deſcribing CD, or the Time in which the Vertex C is preſſed in, to D, by the Stroke.*

Now it is the ſame Thing, whether we ſuppoſe one Globe, moving with Velocity c, againſt another at reſt, or two Globes moving contrary Ways againſt one another, each with the Velocity $\frac{1}{2}c$; or whether we ſuppoſe one Globe moving againſt an immoveable Obſtacle, with Vel. $\frac{1}{2}c$.

By Mechanics, when the Body is given, $v\dot{v} \propto$ Force $\times \dot{x}$, (ſee Prob. I.); but in a falling Body $2s$ is the Velocity, and $\dot{x}$ its Fluxion. And in the Colliſion of the Balls, the Force at D is $\frac{x}{a}p$; therefore $2s\dot{x} : w\dot{x} :: -v\dot{v} : \frac{px}{a}\dot{x}$, and $-v\dot{v} = \frac{2psx\dot{x}}{aw}$, and the Fluent $-v^2 = \frac{2spx^2}{aw}$, but at C, $v = \frac{1}{2}c$, $x = 0$, corrected

FIG. 250. corrected $\frac{1}{4}cc - vv = \frac{2spx^2}{aw}$. And at laſt when C is in B, $v = 0$, and $x = a$, whence $\frac{1}{4}cc = \frac{2sp}{aw} \times aa$, or $cc = \frac{8sp}{aw} \times aa$; therefore, ſince $8s$ and w are given, and $\frac{p}{a}$ is a given Ratio, $cc \propto aa$, or $\frac{c}{a}$ is a given Ratio. Alſo by Reduction $p = \frac{ccw}{8as}$; which put for p gives $\frac{1}{4}cc - vv = \frac{2sxx}{aw} \times \frac{ccw}{8as} = \frac{ccxx}{4aa}$, which reduced is $vv = \frac{cc}{4aa} \times \overline{aa - xx}$.

Again $\dot{t} \propto \frac{\dot{x}}{v}$, and in falling Bodies $\dot{t} = \frac{\dot{x}}{2s}$ (ſee Prob. I.) therefore $\frac{\dot{x}}{2s}$ (Time) : $\frac{\dot{x}}{2s}$ $(\frac{\text{Space}}{\text{Vel.}})$ :: $\dot{t} : \frac{\dot{x}}{v}$, whence $\dot{t} = \frac{\dot{x}}{v} = \frac{2a\dot{x}}{c\sqrt{aa - xx}}$, and (by Form 10) $t = \frac{2a}{c} \times$ Arch, whoſe Sine is $\frac{x}{a}$, and $rad = 1$. But when $x = a$, that Arch $= 90°$; and $t = \frac{2a}{c} \times \frac{3.1416}{2}$; after this Time, the Globes are relatively at reſt; and after an equal Time the Globes are ſeparated again. Therefore $2t$ or $\frac{2a}{c} \times 3.1416 =$ Time of the Contact of the two firſt Globes; during this Time the Motion is propagated the Length of 1 Diameter; whence $(2t)\ \frac{2a}{c} \times 3.1416 : d :: 1'' : \frac{dc}{2a \times 3.1416}$ the Velocity of the Motion in a Second.

It is here ſuppoſed that the Action of the firſt Ball or Globe *A* upon the Second, is over before it affects the third Ball. But if at the ſame Time the ſecond alſo acts upon the third, and the third upon the fourth, *&c.* then the Motion will be propagated further, but then the Time will be longer, which comes to the ſame Thing. But the Time of Contact of any two Balls is ſo very ſmall, that the Action of the firſt upon the ſecond is over, or very near over, before it reaches the third. If the firſt Ball acted ſenſibly upon ſeveral at once, it would be reflected, which is contrary to Experience. FIG. 250.

Cor. 1. *Hence the Velocity of the Motion is the ſame, whatever be the Velocity of the firſt Globe impinging upon the ſecond.*

For d is given, and $\frac{c}{a}$ is a given Ratio.

Cor. 2. *Hence alſo, ceteris paribus, the Velocity of the Pulſes is as the Diameters of the Globes.*

Cor. 3 *In Globes of different Matter, the Value of* a *muſt be found, and then the conſtant Ratio* $\frac{c}{a}$; *whence the Velocity of the Pulſes is had.*

PROB. XXIII.

To find the Velocity of Sound.

Sound is cauſed by the vibrating Motion of a tremulous Body, which excites a Motion in the Parts of the Air that lie next, and theſe excite the like Motions in the Parts that lie next to them, and ſo on. For the Parts of the ſonorous Body alternately going and returning, do in going drive before them thoſe Parts of the Medium that lie neareſt, and by that Impulſe, compreſs and condenſe them, and in return- 251.

FIG. 251. ing, ſuffer them again to recede and expand themſelves. Therefore the Parts of the Medium are condenſed and expanded by Turns at each Vibration; and move to and fro, in like Manner as the Parts of the vibrating Body do; and conſequently the Particles of the Medium have the ſame Motion ſucceſſively communicated to them, and the ſame Laws of Motion, that the tremulous Body has. Now it is a known Property of the Motion of a vibrating Body, that the Force it exerts in vibrating, is as the Diſtance from the middle Point of the Vibration, ſee Prob. II. and in any elaſtic Body, the Force is as the Diſtance it is ſtretch'd to. Therefore each Particle of the Air is acted on with a Force which is alſo as its Diſtance from the middle Point of its Vibration, this being premiſed,

Let ſome certain Space BC be the Diſtance of two ſucceeding Pulſes, Ee the Space thro' which a Particle vibrates; O its middle Point. EG a very ſmall linear Part of the Medium, ſucceſſively transfer'd into the Places $\varepsilon\gamma$, eg, and back again. Then,

Put $BC=b$, EO or $Ce=a$, $EG=p$, $t=$ Time of deſcribing $E\varepsilon$, $\tau=$ Time of deſcribing $G\gamma$, $v=$ Velocity of E at ε, $n=$ Force acting at E when it begins firſt to move, $O\varepsilon=x$, the Diſtance of ε from its Center; and $y=$ Diſtance of γ from its Center at the ſame Time; $A=$ height of a uniform Atmoſphere, $c=3{\cdot}1416$, $s=16\frac{1}{12}$ Feet.

By the Law of Vibration $a : n :: x : \frac{nx}{a} =$ Force at ε, and by Mechanics (ſee Prob. I.) $v\dot{v} \propto F\dot{x}$, and in a falling Body $v=2s$, and $F=p$, therefore $2s\dot{x} : p\dot{x} :: v\dot{v} : \frac{nx}{a} \times -\dot{x}$, and $v\dot{v} = \frac{-2nsx\dot{x}}{ap}$, and $v^2 = \frac{-2nsxx}{ap}$, and corrected $vv = \frac{2sn}{ap} \times \overline{aa-xx}$, and (putting $q=\sqrt{\frac{2sn}{ap}}$) $v = q\sqrt{aa-xx}$.

Again

FIG. 251.

Again $\dot{t} \propto \frac{\dot{x}}{v}$, and in falling Bodies, $2s\dot{t} = \dot{x}$, therefore $\dot{t} : \frac{2s\dot{t}}{2s} :: \dot{t} : \frac{-\dot{x}}{q\sqrt{aa-xx}} = \dot{t}$, and (by Form 10) $t = \frac{-\text{Arch}}{qa}$, whoſe Radius is a, and Sine x; and by Correction (when $x = a$, that Arch $=$ a Quadrant, then) $t = \frac{\text{Arch}}{qa}$, whoſe Coſine is x, let $z =$ that Arch, then $t = \frac{z}{qa}$, and for the ſame Reaſon $\tau = \frac{\text{Arch}}{qa}$, whoſe Coſine is y. Therefore $t - \tau = \frac{\text{diff. Arches}}{qa}$, whoſe Coſines are x and y. Let $\acute{z} =$ diff. Arches, and $\acute{x} =$ diff. Coſines; and by the Nature of the Circle, $\acute{z} = \frac{-a\acute{x}}{\sqrt{aa-xx}}$, therefore $t - \tau = \frac{\acute{z}}{qa} = \frac{-\acute{x}}{q\sqrt{aa-xx}}$.

As the Particles E, G are ſucceſſively agitated with like Motions, the Pulſe in paſſing from B to C, arrives at E before it comes to G; and the Spaces p, b, paſſed over by the Pulſe, are as the Times, that is (as has been ſhewn), as $\frac{\acute{z}}{qa}$, and $\frac{2ca}{qa}$; therefore $\frac{\acute{z}}{p} = \frac{2ca}{b}$. But $E\varepsilon - G\gamma$ or $EG - \varepsilon\gamma = y - x = \acute{x}$, and $\varepsilon\gamma = p - \acute{x}$. Now ſince the elaſtic Force of the Air is as the Denſity, and the Denſity reciprocally as the Space taken up; if $d =$ the mean Denſity, then

FIG. 251. $\frac{1}{p} : d :: \frac{1}{p-\acute{x}} : \frac{pd}{p-\acute{x}}$ or $d + \frac{d\acute{x}}{p} = d + \frac{d\acute{z}}{ap}\sqrt{aa-xx} = d + \frac{2cd}{b}\sqrt{aa-xx}$, the Denſity at $\varepsilon\gamma$. But the mean elaſtic Force : is to p (the Weight of the Lineola p) : : $A : p$; therefore the mean elaſtic Force $= \frac{pA}{p} = A$. And $d : A :: d + \frac{2cd}{b}\sqrt{aa-xx} : A + \frac{2cA}{b}\sqrt{aa-xx} =$ the elaſtic Force in the Place $\varepsilon\gamma$; and its Fluxion is $\frac{2cA}{b} \times \frac{-x\dot{x}}{\sqrt{aa-xx}}$, the Fluxion of the Force at ε correſponding to $\dot{x}$; and $\frac{2cA}{b} \times \frac{-x\acute{x}}{\sqrt{aa-xx}}$ or $\frac{2cAx\acute{z}}{ab}$ is the Decreaſe of the Force correſponding to $\acute{x}$, at the ſame Moment of Time: that is, $\frac{2cAx\acute{z}}{ab}$ or $\frac{4ccAxp}{bb}$ is the Exceſs of the Force at ε above that at γ: and this is the Force by which $\varepsilon\gamma$ is accelerated, and therefore that Force is as x; and when $x = a$, that Force is $\frac{4ccapA}{bb} = n$. Therefore $t = \frac{z}{aq} = \frac{z}{a}\sqrt{\frac{ap}{2sn}} = \frac{z}{a}\sqrt{\frac{bb}{2s \times 4ccA}} = \frac{bz}{2ca\sqrt{2sA}}$. And when $x = a$, then $z = \frac{ca}{2}$, and $t = \frac{cab}{4ca\sqrt{2As}} = \frac{b}{4\sqrt{2As}}$, and $4t = \frac{b}{\sqrt{2As}}$ the Time of vibrating from E to e, and back again, or the Time of the Pulſe running thro' BC. Therefore,

FIG. 251.

fore, $\frac{b}{\sqrt{2As}}$ (Time) : b (Space) : : $\overset{''}{1}$: $\frac{b\sqrt{2As}}{b} =$ $\sqrt{2As}$, the Velocity of the Pulſes in a Second.

At a mean Gravity of the Air, when the Quickſilver ſtands at 30 Inches, the ſpecific Gravity of Air to Quickſilver is about as 1 to 11890, therefore the Height of a uniform Atmoſphere muſt then be 29725 Feet $= A$. Therefore $\sqrt{2As} = \sqrt{32\frac{1}{6} \times 29725} =$ 978 Feet, the Velocity *per* Second.

But the Velocity of ſound is really greater; for here is no Allowance made for the Diameters of the Particles of Air and Vapour, which are unknown. For as theſe ſolid Particles themſelves are incompreſſible, the Velocity is propagated inſtantaneouſly thro' them; and therefore the Space found before ſhould be augmented by a Quantity which is to that Space, as the Diameters of the Particles to their Diſtances: And ſo we find by Experience, that Sound moves at the Rate of about 1140 Feet in a Second, at a mean Denſity of the Air.

Cor. 1. *Hence the Velocity of Sound is always the ſame in the ſame Medium, whether it be more or leſs intenſe; for the Velocity is* $\sqrt{2As}$, *in which* a *is not concern'd.*

Cor. 2. *Hence alſo the Diſtances of the Pulſes are known, by dividing* 1140 *by the Number of double Vibrations made in a Second. Thus by Schol. Prob. II. in a String that makes* 300 *Vibrations in a Second,* $\frac{1140}{150}$ $= 7\frac{3}{5}$ *Feet, the Breadth of one Pulſe.*

Cor. 3. *The Velocity of Sound* (*ceteris paribus*), *is the ſame on the Top of a Mountain, as in a deep Valley.*

For if the incumbent Weight of the Atmoſphere be increaſed or decreaſed in any Ratio; its Denſity (and the Height of the Mercury) will be increaſed or decreaſed in the ſame Ratio, and the Height A, of a uniform Atmoſphere of that Denſity, will continue the ſame. And therefore,

Cor.

FIG. Cor. 4. *The Temper of the Air remaining the ſame, as to heat and cold; the Velocity of Sound will be the ſame, how much ſoever the Air be condenſed or rarified.*

SCHOLIUM.

252. To explain more particularly the Motion of the Particles of Air, when condenſed and expanded as the Pulſes move thro' them; let *B*, *C*, &c. be certain Waves or Pulſes of the Air moving from *B* towards *C*; draw the Line *KF*, and *ST* parallel to it, and at any Place *Q*, erect *Qs*, for the Denſity of the Medium at *Q* or *r*, *Qr* being the mean Denſity. And ſuppoſe *O* a fixt Point at the Place *F*, and *p* a Particle of the Air exceeding near it; and let the Pulſes of Air, move ſucceſſively thro' the Points *O*, *p*, &c. or rather (for the more eaſy conceiving it) let *O* be ſuppoſed to move with a uniform Motion along the Line *FK*, thro' the Waves *C*, *B*, &c. at reſt. It is plain the Particle *p*, going with it, cannot move with a uniform Motion, by Reaſon of the alternate Contraction and Dilatation of the Air, but will move ſometimes to one Side, ſometimes to the other Side of *O*, if *Op* be of a due Quantity. Now whilſt *O* moves from *F* towards *C*, the Medium growing denſer, *p* will approach nearer to *O*, and at *C* where the Denſity is greateſt, *p* and *O* will coincide; and proceeding forward from *C* to *D*, the Denſity ſtill being greater than the Mean, *p* will ſtill loſe Ground, and at *D*, the Place of mean Denſity, *p* will be the furtheſt behind *O*; and in paſſing from *D* to *H*, into a rarer Medium, the Air expanding towards *H*, *p* will move faſter than *O*; and at *H*, (the Place of leaſt Denſity), they will coincide. In proceeding to *I*, the Medium ſtill continuing rarer, *p* will ſtill move faſter than *O*; and at *I* where the mean Denſity is,) *p* will be then the furtheſt before *O*, the ſame as it was at *F*. And after this Manner the Points *O*, *p*, will go thro' all the Waves or Pulſes; the Particle *p* vibrating firſt

to one Side of O, and then to the other. And the Phænomena will be the ſame, if O ſtands ſtill, and the Waves move. The Points O and p, or the Syſtem Op repreſents OE (in Fig. 251.) Now we may obſerve that when the Syſtem is at r, it is in a denſer Part of the Wave than it was at F, where it begun to move, that is (Fig. 251.) the Denſity at $\varepsilon\gamma$ is greater than at EG; for the ſame Particles of Air are in EG and $\varepsilon\gamma$ at different Times, and in different Parts of the Wave F, and r. But ε, γ, being contemporary Poſitions of the Points, E, G; the Denſity at ε is greater than at γ, as is plain, by Fig. 252. Therefore the greater Denſity at ε will move $\varepsilon\gamma$ towards O. FIG. 252.

And to conſtruct the *Curve of Denſity FCD*; with the Radius OE or $Oe = a$, deſcribe the Circle EVe, take $Os = x$, and erect the Ordinate $sV = u$. Let Arch $EV = z$, mean Denſity $= 1$, $FI = b$. Then in FK, take $Fr = \frac{bz}{2ca}$, and at r erect $r\varepsilon = \frac{2cu}{b}$, and the Point ε is in the Curve, and $r\varepsilon$ is the Exceſs (or Defect) of Denſity at r. And for the Nature of the Curve; put $Fr = X$, $r\varepsilon = Y$, then $\frac{b}{2c} = \frac{aX}{z} = \frac{u}{Y}$, whence $aXY = zu$, a mechanical Curve.

PROB.

PROB. XXIV.

FIG. 253. *A Cord or ſmall Rope being fixt at one End, and extended by Means of a Weight at the other End; if the Rope be ſtruck near one End, the Stroke will excite a Motion in it, which will run along the Rope in Form of a Wave. To find the Velocity of the Motion.*

We will ſuppoſe the Cord to be exceeding ſlender, or rather a heavy Line, perfectly flexible; let *AB* be the Line, and ſuppoſe the Point *D* to be impelled as far as to *d* by the ſtroke; draw *adb* parallel to *ADB*. As ſoon as the Point *D* arrives at *d*, the Line *DE* will be found in the Poſition *dSE*. For the Force being impreſſed upon the Particles of the Curve at *D*, and theſe put into Motion, they will exert their Force upon the Parts next to them, and they upon the next, and ſo on; ſo that the firſt Motion will be ſucceſſively communicated to all the Parts of the Cord, and they will all, one after another, be carry'd over to the Line *ab*; where all their Motion will be deſtroyed; and they will remain there, if *D* remains at *d*. In the mean Time the Curve Line (or Wave) *dSE*, will be carried along to *gCF*, and ſo forward.

As none of the Parts of the Curve beſides thoſe at *D*, received any ſtroke; it is plain they are all moved by means of the Cord, communicating *D*'s Motion to all the reſt in order. And ſince the Force which the Cord exerts on any Particle (by Cor. Ex. 16. Prop. XIII.) is as the Curvature there; therefore the Curve near *F* is convex towards *GF*. Likewiſe

FIG. 253.

wiſe ſince the Particles of the Cord, after they have received their Motion, are not carried beyond the Line *ab*, but ſtop at it; it is plain that near *g*, the Curve is convex towards *gb*. It is alſo evident that at any two Points (*n*, *o*) of the Curve *gCF*, that are equally diſtant from the Lines *AB*, *ab*; the Curvature will be the ſame. For as the Motion of any Particle is generated gradually as it moves from the Line *AB*; it muſt be gradually retarded in approaching *ab*, ſo as to loſe all its Motion there; therefore the Forces, and conſequently the Curvature at *o* and *n*, are equal and contrary, and at the Middle *C* is nothing: And ſo each Particle is accelerated as far as *C*, and afterwards retarded. Moreover the Parts of the Rope as they are drawn from *F* towards *C*, will be drawn nearer into a right Line as they approach to *C*; ſo the Curvature will continually decreaſe to *C*, the Point of contrary Flexure, and therefore the Force will likewiſe decreaſe towards *C*; and hence the Parts of the Curve, *gC*, *FC* are exactly ſimilar and equal. This being premiſed, the Time is to be found wherein the Wave is carried from *G* to *F*, or which is the ſame, wherein a Particle at *G* is tranſlated to *g*.

Let *Fk* or $kf = a$, $km = x$, $mn = y$, $Fn = z$, $Ck = d$, $r =$ Radius of Curvature in *F*. $c = 3.1416$, and let the Force at *n* be as *x*. By Prob. V. Sect. II. If $\dot{z}$ be given, the Radius of Curvature at *n* is $\frac{\dot{z}\dot{x}}{-\ddot{y}}$ or $\frac{\dot{z}\dot{x}}{\ddot{y}}$ (becauſe *x* decreaſes). Then (by Cor. Ex. 16. Prop. XIII.) $x : a :: r : \frac{\dot{z}\dot{x}}{\ddot{y}}$, and $ar\ddot{y} = \dot{z}x\dot{x}$, and the Fluent is $ar\dot{y} = \frac{1}{2}\dot{z}xx$, but at *F*, $\dot{y} = \dot{z}$, and $x = a$, therefore $ar \times \overline{\dot{y} - \dot{z}} = \frac{1}{2}\dot{z} \times \overline{xx - aa}$; and $2ar\dot{y} = \overline{2ar + xx - aa} \times \dot{z}$, and $4aarr\dot{y}^2 = \overline{2ar + xx - aa}^2 \times \overline{\dot{y}\dot{y} + \dot{x}\dot{x}}$; and $4ar \times \overline{xx - aa} \times \dot{y}^2 = \overline{2ar + xx - aa}^2 \times$

FIG. $\dot{x}^2 = 4aarr\dot{x}\dot{x}$ nearly, becauſe a is extreamly ſmall in
253. reſpect to r; therefore $\dot{y} = \frac{\dot{x}\sqrt{ar}}{\sqrt{aa - xx}}$; and (by Form 10) $y = \frac{\sqrt{ar}}{a} \times$ Arch, whoſe Radius is a, and Sine x; but when $x = a$, $y = 0$, that Arch is a Quadrant; whence $y = \frac{\sqrt{ar}}{a} \times$ Arch whoſe Coſine is x, and when x becomes $= 0$, that Arch is a Quadrant $= \frac{ca}{2}$, and $y = d$, therefore $d = \frac{1}{2}c\sqrt{ar}$, whence $r = \frac{4dd}{cca}$; therefore the Rad. Curvature in n is $\frac{4dd}{ccx}$.

To find the Time of moving from F to m. Let $t =$ the Time, $v =$ Velocity in m, $h = 16\frac{1}{12}$ Feet, $l =$ Length of the Cord, $w =$ its Weight, $p =$ its Tenſion; then $\frac{w\acute{z}}{l} =$ Weight of the Particle $\acute{z}$. Then (by Ex. 16. Prop. XIII.) $\frac{4dd}{ccx} : \acute{z} :: p : \frac{pccx\acute{z}}{4dd} =$ Force at m or n. And by Mechanics (ſee Prob. I.) $v\dot{v} \propto F \times \dot{s}$; that is, $2h \times \dot{s} : \frac{w\acute{z}}{l} \times \dot{s} :: v\dot{v} : \frac{pccx\acute{z}}{4dd} \times -\dot{x}$, and $v\dot{v} = \frac{-2hpcclx\dot{x}}{4ddw} = -Bx\dot{x}$ (by Subſtitution), and $vv = -Bxx$, and by Correction, $vv = B \times \overline{aa - xx}$.

Again, $\dot{t} \propto \frac{\dot{x}}{v}$; or $\frac{\dot{s}}{2h}\ (\dot{t}) : \frac{\dot{s}}{2h}\ (\frac{\dot{x}}{v}) :: \dot{t} : \frac{-\dot{x}}{v}$; or $\dot{t} = \frac{-\dot{x}}{\sqrt{B}\sqrt{aa - xx}}$. And $t = \frac{1}{\sqrt{B}} \times \frac{\text{Arch}}{a}$ whoſe Radius is a, and Sine x; and by Correction,

$t =$

FIG. 253.

$t = \frac{1}{\sqrt{B}} \times \frac{\text{Arch}}{a}$, whoſe Coſine is x; and when $x = 0$, that Arch is $= \frac{ca}{2}$; therefore $t = \frac{ca}{2\sqrt{Ba}} = \frac{c}{2\sqrt{B}} = \frac{c}{2} \times \sqrt{\frac{4ddw}{2hpccl}} = d\sqrt{\frac{w}{2hpl}}$, the Time of a Particle moving thro' Fk, or of the Wave moving from C to k, or thro' $\frac{1}{2}GF$ or d; whence $d\sqrt{\frac{w}{2hlp}}$ (Time) : d (Space) : : $\overset{\prime\prime}{1}$: $\sqrt{\frac{2hlp}{w}}$, the Velocity of the Wave in a Second.

And ſince t and y are each of them as the Arch of the Coſine x; therefore t is as y; and y increaſes uniformly, as it ought; for any Point of the Curve advances forward equal Spaces in equal Times. Therefore the Force is as x, as was ſuppoſed; and all the Requiſites are truly found, and the Problem rightly ſolved, and the Curve is ſuch as is here deſcribed.

Cor. 1. *Hence in the ſame Cord and Tenſion, the Motion will be equally ſwift, whether the impreſſed Force at D be greater or leſſer; that is, greater Waves and leſſer ones have the ſame Velocity.*

Cor. 2. *The Pulſe is ſwifter when the Tenſion is greater, and that as the ſquare Root of the tending Force.*

Cor. 3. *When the Pulſe has run from one End to the other, it will be returned back again in an equal Time: And thus it will run back and forward, till the Motion be ſpent.*

For the fixt Point at the End cannot obey the Motion of the Wave; and by its Reſiſtance it is equivalent to a new Impulſe.

Cor. 4. *If the Cord is not perfectly flexible, but rigid and ſtiff; the Velocity will be greater in the ſub-duplicate Ratio of the Degree of Stiffneſs.*

For the greater Stiffneſs ſupplies the Place of a greater Tenſion.

FIG. 253. COR. 5. *If ſeveral Strokes be made ſucceſſively at D; or, which is the ſame Thing, if the End A be ſhaken backward and forward; there will be generated ſo many Waves gradually ſucceeding one another, as P, Q, R, S, and the Diſtance of two Waves will be known by dividing* $\surd\frac{2hlp}{w}$, *by the Number of Strokes or Shakes in a Second.*

COR. 6. *The Wave or Pulſe will run thro' the Length of any Part of the Cord, in the ſame Time as that Part would perform one Vibration.*

This appears from Prob. II. compared with this.

SCHOLIUM.

And to conſtruct the Curve or Figure of the Wave. With the Radius *Fk* (or *a*) deſcribe the Quadrant of a Circle, *FTV*; make $km = x$, draw the Ordinate *mT*, and erect the Ordinate $mn = \surd\frac{r}{a} \times$ the Arch *FT*, or, which is the ſame, $mn = \frac{2d}{ca} \times$ Arch *FT*. And hence it appears, that this * *funivolant Curve* is the ſame as the harmonical Curve, conſtructed in Schol. Prob. II. For (Fig. 218.) $AE = \frac{ds}{b}$, and $AZ = \frac{d}{b} \times$ Quadrant, and $AZ - AE$ or $ZE = \frac{d}{b} \times$ Arch, whoſe Coſine is *s*. And both theſe Curves are of the ſame Nature as the Curve of Denſity, conſtructed in the Scholium of the laſt Problem; and all of them are related to the Figure of Sines.

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* Scœnobatical.

PROB.

PROB. XXV.

To find the Height of the Tides.

FIG. 254.

Let R = *Radius of the Earth's Orbit, or the Diſtance of the perturbating Body,* ⊙*C.*
p = *periodical Time of the Earth round the Sun, in Seconds.*
g = *Gravity of any Parcel of Matter, as* 1 *Foot.*
h = 16.1 *Feet, the Space deſcended in* $1''$ *by Gravity.*
$\pi = 3.1416$.

By uniform Motion, $p : 1'' :: 2\pi R : \frac{2\pi R}{p} = A$, the Arch deſcribed in 1 Second by the Earth; and $\frac{AA}{2R}$ or $\frac{2\pi\pi R}{pp}$ = verſed Sine of that Arch. And (the Forces being as the Spaces deſcended thereby, in a given Time; therefore) $h : g :: \frac{2\pi\pi R}{pp} : \frac{2\pi\pi gR}{pph}$ the centripetal Force of the Sun, or the perturbating Body at C. But by the Theory of Gravity, the Radius R is to $3CD$; as the centripetal Force at C, to the perturbating Force at M or D; that is, $R : 3CD :: \frac{2\pi\pi gR}{pph} : \frac{6\pi\pi g}{pph} \times CD$ or $f \times CD$ = the diſturbing Force of the Sun, *&c.* at D; putting $f = \frac{6\pi\pi g}{pph}$.

Or thus,

Let q = *Radius of the Earth.*
e = *its Denſity.*
b = *Semi-diameter of the perturbating Body at* ⊙.

s =

FIG. 254. $s =$ *Sine of its apparent Semi-diameter.*
$d =$ *its Denſity.*

Then the Forces of Bodies being as the Quantities of Matter divided by the Squares of the Diſtances; and the Matter, as the Denſity $\times d$ by the Cube of the Radius, we have $\frac{a^3e}{aa} : g :: \frac{b^3d}{RR} : \frac{b^3dg}{RRae} =$ Force of the perturbating Body at *C*. And by the Theory of Gravity, $R : \frac{b^3dg}{RRae} :: 3CD : \frac{3dgb^3}{aeR^3} \times CD$, the perturbating Force at *D* or *M*. But by Trigonometry $\frac{b}{R} = s$; therefore $\frac{3dgs^3}{ae} \times CD$ or $f \times CD =$ perturbating Force of the Body at *M* or *D*, putting $f = \frac{3dgs^3}{ae}$.

Let *APBQ* be the Earth, $CA = a$, $CP = b$, $CD = y$, $CE = x$, then the diſturbing Force at $D = fy$; and the Force of Gravity at $D = \frac{gy}{a}$, and at $E = \frac{gx}{b}$; therefore the Gravity or Force of *D* towards $C = \frac{gy}{a} - fy$. Now the whole Columns *AC*, *PC*, being of equal Preſſure at *C*; their Fluxions multiply'd by the gravitating Forces muſt be equal; that is, $\frac{gy\dot{y}}{a} - fy\dot{y} = \frac{gx\dot{x}}{b}$, and the Fluent $\frac{gyy}{2a} - \frac{fyy}{2} = \frac{gxx}{2b}$, or $\frac{yy}{a} - \frac{f}{g}yy = \frac{xx}{b}$, and when $y = a$, and $x = b$, then $a - \frac{f}{g}aa = b$, and $a - b = \frac{f}{g}aa$.

In Caſe of the Sun, $a = 21000000$, $p = 31557600$, then $a - b = \frac{f}{g}aa = \frac{6\pi\pi aa}{ppb} = 1.63$ Feet $= 19\frac{1}{2}$ Inches the Height of the Solar Tide.

In

FIG. 254.

In Caſe of the Moon, $s = S.15'\frac{2}{3}.d = 11$, $e = 9$, and $a - b = \frac{f}{g}aa = \frac{3ds^3a}{e} = 7.28$ Feet, the Height of the lunar Tide. And the Height by their joint Force, when in Conjunction, is 8.91 Feet.

And the ſame Forms will ſerve for finding the Height of the Tides on any other Planet, or celeſtial Body; *mutatis mutandis.*

SCHOLIUM.

The Height of the Tides here given is ſuppoſed to be at ſuch Places where the Sun and Moon are vertical, and alſo in the Equinoctial. In Places at a Diſtance from the Equator, the Height will be leſs, as the Latitude is greater; their Height will alſo be leſs, according to the Sun or Moon's Declination from the Equator.

PROB. XXVI.

To find the Preceſſion of the Equinoxes. 255.

Let $ApEP$ repreſent the Earth, orthographically projected upon the Plain of the ſolſtitial Colure, PCp the equinoctial Colure; P, p the Poles, AE the Equinoctial, IK a parallel of Latitude, BCD the Ecliptic; let ☉ be the Sun at an immenſe Diſtance, and vertical to D, in the Tropic, draw $QCq \perp BD$.

The Earth being an oblate Spheroid or nearly ſuch, the Sphere $PApE$ is encompaſſed with a ſolid Cruſt, this Cruſt is ſpread all over the Sphere, and goes round the Equator in Manner of a Ring, whoſe Thickneſs there is Aa or Ee; in Places at a Diſtance from the Equator, as at I, it is thinner. Now the

Preceſſion

FIG. 255. Preceſſion of the Equinoxes ariſes from the Action of the Sun and Moon, upon this Cruſt or exterior Part of the Earth; the Part *PApa* being attracted toward them, in Lines as *AG* parallel to $C☉$; and the other Side *PEpe* acted on the contrary Way, or in the Direction *FE*; and both conſpiring to turn the Earth about, in the order of the Letters *BPQAD*; while the inner Sphere is acted on all Sides alike. Now the Effect of theſe Forces upon this Cruſt, and the Motion communicated thereby to the whole Body of the Earth, is what we are now to enquire after; and we ſhall firſt begin with the Sun's Force.

1. Take Arch $XL=Xl$, and draw *LM*, *XY*, *lm* parallel to *DB*; and *XN*, *ln* parallel to *Qq*; and put $p=$ periodical Time of the Earth about the Sun; $g=$ Gravity of any Parcel of Matter, as 1 Foot; $h=16.1$ Feet, the Space deſcended in a Second by Gravity, $\pi=3.1416$; then, by what was demonſtrated in the laſt Problem, we have the diſturbing Force of the Sun $=f\times LM=\dfrac{6\pi\pi g}{pph}\times LM$; alſo let the mean Radius of the Earth $=a$, $Aa=m$, $IX=r$, $CX=v$, $S.\angle DCA=s$, Coſ. $DCA=c$; Arch XL or $Xl=z$, Sine XL or $Xl=y$, then will $NX=sy$, $NL=cy$, $XY=sv$, and $CY=cv$. Therefore $LM=cy+sv$, $CM=cv-sy$, $lm=sv-cy$, $Cm=cv+sy$.

Now the Force of two Parcels of Matter at *L* and *l*, to wheel the Earth about the Center *C*, is $=CM\times f\times LM+Cm\times f\times lm$; the former acting by the Power of the Lever *CM*, the other by that of *Cm*: that is, the Force of theſe two Parcels of Matter is $=\overline{cv-sy}\times\overline{cy+sv}+\overline{cv+sy}\times\overline{sv-cy}:\times f,=2csf\times\overline{vv-yy}$; therefore the Fluxion of the Force of all the Matter in the Circumference *IK* is $=2csf\dot{z}\times\overline{vv-yy}$, this Force is directed from *q* towards *p*, *D*; therefore the Flux. of the Force the contrary Way, or in Di-

rection

FIG 255.

rection *QAD* is $= 2csf\dot{z}\times yy-vv = 2csf\times\frac{ry^2\dot{y}}{\sqrt{rr-yy}} - 2csfvv\dot{z}$; whoſe Fluent (by Form 10 and 11) is $= csrrfz - csrfy\sqrt{rr-yy} - 2csfvvz$, and when $y=r$, and doubled, the Force of the Circumference *IK* is $= \pi csfr\times\overline{rr-2vv} = \pi csfr\times\overline{aa-3vv}$. *Therefore if* $v=\frac{1}{3}a$, *the Force of the Annulus IK is nothing.*

2. By the Nature of the Ellipſis, $a:r::m:\frac{mr}{a}$ = Height of the Matter at the Circumference *IK*; therefore the Fluxion of the Force in the Annulus *IK* is $= \pi csfr\dot{v}\times\overline{aa-3vv}\times\frac{mr}{a} = \frac{\pi csfm\dot{v}}{a}\times\overline{aa-3vv}\times\overline{aa-vv} = \frac{\pi csfm\dot{v}}{a}\times\overline{a^4-4aavv+3v^4}$, and the Fluent $= \frac{\pi csmf}{a}\times\overline{a^4v-\frac{4aav^3}{3}+\frac{3v^5}{5}}$; and when $v=a$, and doubled, the Force of the whole Cruſt $= \frac{8\pi csma^4f}{15} = \frac{16}{5}\times\frac{\pi^3csma^4g}{pph}$; *therefore the Force is as the Thickneſs of the Cruſt*; and this is equal to the ſimple Force $\frac{16\pi^3csma^3}{5pph\sqrt{\frac{2}{5}}}g$, acting at the Diſtance $a\sqrt{\frac{2}{5}}$ from the Center *C*, or by the Power of the Leaver $a\sqrt{\frac{2}{5}}$. And this is the Force with which the Sun diſturbs the Earth, when in the Tropic at *D*; call this ſimple Force *Fcs*.

3. But we muſt find what this Force will be when the Sun is in any other Point of the Ecliptic as at *H*. Draw the great Circle *pHRP*, and making Rad. $=1$, let Arch $CH=z$, Sine $CH=y$, then in the Spherical Triangle *CHR*, right angled at *R*, Rad. (1) : *S.CH* (y) : : *S.C* (s) : $S.HR=sy$; therefore ſince *F* conſiſts only of given Quantities, the Force of the Sun at *H* is $= Fsy\sqrt{1-ssyy}$; for it was found in the laſt Article to be $= F\times$ by the Rectangle of the Sine and

FIG. 255. Coſine of the Sun's Height above the Plane of the Equinoctial. But this very ſmall Force $Fsy\sqrt{1-ssyy}$, acts altogether in the Plane *PRHp*, therefore we muſt divide it into two Forces, one acting in the Plane *PQA* which we want; the other in the Plane *PCp*, perpendicular to the other; this latter Force is deſtroyed by an equal and contrary Force, when the Sun is equidiſtant on the other Side of the Tropic; but the other Force acting always one way, is the only one the Earth is annually affected with. The Tan. $HC = \frac{y}{\sqrt{1-yy}}$, and by Sph. Trig. $1 : c :: T.AC \left(\frac{y}{\sqrt{1-yy}}\right) : \frac{cy}{\sqrt{1-yy}} =$ Tan. $RC = t$; and $S.RC = \frac{t}{\sqrt{1+tt}}$

$$= \frac{cy}{\sqrt{1-yy}\sqrt{1+\frac{ccyy}{1-yy}}} = \frac{cy}{\sqrt{1-yy+ccyy}} =$$

$\frac{cy}{\sqrt{1-ssyy}}$, then to find the Part of the Force acting in the Plane *PQA*, it will be, Rad. $(1) : Fsy\sqrt{1-ssyy}$ (whole Force) :: $S.RC \left(\frac{cy}{\sqrt{1-ssyy}}\right) : Fcsyy$, the Force in Direction *PQ*: *Therefore this Force is as the Square of the Sun's Diſtance from the Equinox.* From hence to find the mean annual Force, we muſt firſt find the Sum of all the $Fcsyy$ in the Circle, or the Fluent of $Fcsyy\dot{z} = \frac{Fcsyy\dot{y}}{\sqrt{1-yy}}$, and the Fluent (by Form 10, 11) is $= \frac{Fcsz}{2} - \frac{Fcsy}{2}\sqrt{1-yy}$, and when $y=1$, the whole Fl. $= \frac{Fcs\pi}{4}$, and in the whole Circle $= Fcs\pi$; this divided by the whole Circumference 2π, the mean Force is $= \frac{1}{2}Fcs$, *which is but half the greateſt Force, when the Sun was in the Tropic.* FIG. 255.

4. Put $n =$ Time of the Earth's diurnal Rotation, and ſuppoſing it of uniform Denſity, its Solidity is $= \frac{4\pi a^3}{3}$. And by Ex. 6. Prob. XXIII. Sect. II. the Diſtance of the Center of Gyration, from the Center *C* of the Earth is $= a\sqrt{\frac{2}{5}}$, and by the Property of the Center of Gyration, if the whole Matter of the Earth was ſuppoſed to be collected into that Point, any Force apply'd to move it about the Center *C*, would generate the ſame angular Velocity in it, and in the ſame Time, as it would do in the Earth itſelf. And ſince this Force $\frac{1}{2}Fcs$, acts at the ſame Diſtance $a\sqrt{\frac{2}{5}}$, therefore it is the ſame Thing as if that Force was directly apply'd to the Body to move it: Now to find the Motion generated herein. By Mechanics the Space deſcribed or generated in a given Time, is as the Force directly, and Matter reciprocally; therefore : $\frac{g}{1} : b :: \frac{\frac{1}{2}Fcs}{\frac{4}{3}\pi a^3} : \frac{3Fcsb}{8\pi a^3 g}$ $=$ Space deſcribed in $1''$, by the Matter of the Earth in the Center of Gyration. And $2\pi a\sqrt{\frac{2}{5}}$ (circumf.) $: 360^\circ :: \frac{3Fcsb}{8\pi a^3 g} : \frac{3Fcsb}{16\pi^2 a^4 g\sqrt{\frac{2}{5}}} \times 360 = \frac{3cs\pi m}{2app} \times 360$, the Angle deſcribed about the Earth's Center, by the Center of Gyration, which is equal to the angular Motion of the Earth in $1''$.

5. In the Equator *AE*, make $Ab = \frac{360}{n}$, and perpendicular to it make $bt =$ Earth's angular Motion juſt found. Now ſince in $1''$ of Time any Point *A*, is carry'd from *A* to *b* by the Earth's Rotation; and in the ſame Time the Point *b* of the Equinoctial is moved to *t*, by the Sun's diſturbing Force; its plain,

FIG. 255. any Point *A* will purſue the Tract *Atr*; that is, the Equinoctial Point will be moved from *C* to *r*, in 1″. To find which, in the Triangle *Abt*, *Ab* or *At* $\left(\frac{360}{n}\right)$: Rad. (1) : : *bt* $\left(\frac{3cs\pi m}{2app} \times 360\right) : \frac{3cs\pi mn}{2app} =$ S. ∠ *bAt*, or Arch *ro*; and in the Triangle *roC*, S.*C* (*s*) : S.*ro* $\left(\frac{3cs\pi mn}{2app}\right)$: : Rad. (1) : S.*rC* $= \frac{3c\pi mn}{2app} =$ Sine of the Preceſſion of the Equinoxes, by the Sun's Force, in 1″; and $\frac{3c\pi mn}{2ap} =$ S. Preceſſion in a Year. *Therefore the Preceſſion is as the periodic Time of the Earth's Revolution about its Axis.*

6. Laſtly (by Prop. 37. L. III. *Newton's Principia*) the Sun's Force is to the Moon's, as 1 to 4.4815; and the Sum of both is $5.4815 = q$, therefore the Sine of the Preceſſion in a Year by both Sun and Moon is $= \frac{3c\pi mnq}{2ap}$, where $c = .91706$, $m = 89460$ Feet, by Prob. XVIII. $n = 86160''$, $a = 21000000$, and $p = 31557600$; whence $\frac{3c\pi mnq}{2ap} = ,0002755$. But ,0002909 is the Sine of 1 min. or 60″; therefore 2909 : 60″ : : 2755 : 56.83″, the annual Preceſſion of the Equinoxes. Which being more than it is found to be by Obſervation, makes it probable, that the Earth is more denſe towards the Center; or perhaps, that the Moon's Force to the Sun's is not rightly adjuſted, being only collected from the riſing of the Tides. But that the Earth is rarer at the Surface appears from this, that the moſt Part of the Surface is Water.

COR. 1. *If the Earth had no Rotation about its Axis, The Inclination of the Equator to the Ecliptic would decreaſe*

FIG. 255.

creaſe above two Degrees and a half in one Year: And conſequently in a few Years, the Equinoctial would fall into the Ecliptic.

For $\overset{\prime\prime}{1} : pp :: \frac{3cs\pi m}{2app} \times 360 : \frac{3cs\pi m}{2a} \times 360 =$ 2.681 Degrees in one Year.

Cor. 2. *The other Force acting in Direction PC perpendicular to PQA,* (which was uſeleſs in finding the Motion of the Equinoxes) *changes its Direction four Times in one periodical Revolution.* And this generates a Motion, which compounded with the Earth's diurnal Motion, *cauſes a Vibration or nodding* of the Earth's Axis, to and from the Ecliptic, *twice in a Year.* But becauſe the Moon does not accompany the Sun thro' a Quadrant of the Ecliptic, we cannot conſider their joint Force; therefore if we regard only the Force of the Sun (for the Effect of the Moon's Force alone will be far leſs); *then the whole Quantity of this Nutation will not amount to a ſingle Second.*

Cor. 3. *But there is a Nutation in the Earth's Axis, cauſed by the Moon's being out of the Ecliptic.* For in the foregoing Computations, we ſuppoſed her always in it; but the Moon's Orbit being inclined to the Ecliptic, in an Angle of about 5 Degrees, ſhe will only be in the Ecliptic when ſhe's in the Nodes; and at all other Times ſhe will have a greater or leſſer Force to move the Earth, according as her Declination is greater or leſſer. And this Force (as was ſhewn Art. 4.) is as the Rectangle of the Sine and Coſine of Declination, that is, as the Sine of twice the Declination: And from hence ariſes a Perturbation of all the former Motions, which will be greateſt when ſhe is furtheſt from the Ecliptic, or 90° from the Node. And as the Nodes make a Revolution in about 18.6 Years, therefore this Force will be directed all manner of Ways during one Period of the Nodes, and will therefore cauſe the Poles of the Equinoctial

FIG. 255. noctial to wander about the mean Pole, deſcribing a circular Figure in the Heavens,. in the ſaid Period of Time.

Now to compute the Quantity of this *Evagation of the Poles*; the Arch rC or $56''.83$ was meaſured along the Ecliptic; but $s \times 56.83$, or $22''.66 = ro = LA$, which meaſures the Motion of the Pole, and that by the joint Forces of both Sun and Moon. Therefore $\frac{4 \times 4815}{5 \times 4815} \times ro = 18''.52$, the Effect of the Moon's Force alone in a Year; and this Force of the Moon in the Ecliptic is as $S.\,47^d \doteq .7313$; and at 5 deg. Lat. when the Nodes are at the Equinoxes, as $S.\,57^d = .8386$; the diſturbing Force then will be $= .8386 - .7313 = .1073$, this is the leaſt Quantity of Force; and the greateſt is as $S.\,10^d = .1736$, or thereabouts, when the Nodes are in the Solſtices. And in other Poſitions this perturbating Force is different; but we cannot be far wrong if we take a Mean between the greateſt and leaſt, which is .1404; whence this Proportion, $7313 : 1404 :: 18''.52 : 3''.55$, the annual Effect of the Moon's diſturbing Force. And $3''.55 \times 18.6 = 66''.17$; which, as the Motion is directed all Ways, muſt be laid out in the Circumference of a Circle; and then $\frac{66.17}{3.1416} = 21''.06$, the Diameter of the *Circle of Evagation*; which would appear diſtinct in the Heavens, if the other Forces and Motions were away. But we muſt obſerve, that if the Earth is denſer within, and more rare at the Surface, the Diameter of this Circle will come out leſs than is here determined.

To find the Direction of this Motion. It was ſhewn in Art. 5. that when the Solſtice is at D or B, the Equinoctial Circle is moved by the Forces of the Sun and Moon, from the Place AC, to the Place Ar; conſequently the Pole P is moved from P towards C; and the Caſe is the ſame with this diſturbing Force of the Moon. For ſuppoſe vw to be the Moon's Orbit, w the nonageſſimal Degree, or higheſt Point of the Orbit elevated above the Plane of the Ecliptic DB, towards the Pole P; then v and w being ſituated at D and B, the Force of the Moon, being now greater, will alſo move the Pole P towards C. And for the ſame Reaſon, when (by the retrograde Motion of the Nodes) w is got to S, P will be moving towards R; when w is at C, P moves towards A, ſo that P always moves the ſame Way, or in the ſame Direction that w does. Therefore about π, deſcribe the ſmall Circle $\alpha\beta\gamma\delta\varepsilon$; and let α be the Place of the Pole when w is at B: Then whilſt w paſſes ſucceſſively thro' BS, SC, CH, HD; the Pole P (moving the ſame Way) will paſs thro' $\alpha\beta$, $\beta\gamma$, $\gamma\delta$, $\delta\varepsilon$; the contemporary Poſitions of w and P being B and α; S,β; C,γ; H,δ; D,ε, *&c.* and therefore the Circle deſcribed $\alpha\gamma\varepsilon$ muſt of Conſequence include the mean Pole π. *And from hence it is plain, that the moving Pole is in every Place (as ſuppoſe at* α*) nearer to* w *the higheſt Point of the Orbit (at B), than the mean Pole* π*, by the Radius of the little Circle* $\pi\alpha$ *or* $10\frac{1}{2}$ *Seconds.*

FIG. 255.

I need ſcarce mention that this perturbating Force, and its Effect, is greateſt in the Moon's Perigee, being nearly in the reciprocal triplicate Ratio of the Moon's Diſtance, as is well enough known.

SCHOLIUM.

By a like Method (but by a Proceſs much more eaſy) the Motion of the *Moon's Nodes* may be computed. And I might now proceed to the Calculation thereof, and of the *Motion of the Apſides*, and other Things belonging to the *Moon's Motion*, and other difficult Problems, ſuch as finding *the Curves de-*

FIG. 255. *ſcribed by Bodies acted upon by any Laws of Gravity, and moving in Mediums which reſiſt, as any Powers of the Velocity, and ſuch like.* But ſince theſe cannot be diſpatched in a few Words, but often run into long and tedious Calculations, and require a great deal of room, I ſhall not trouble the Reader with them; eſpecially ſince the Method of purſuing and managing theſe, is the very ſame as in thoſe Problems here delivered. And therefore I ſuppoſe, if the Reader underſtands what has been before laid down; he will be able of himſelf to apply *this Doctrine*, to the Solution of any other Problem that happens to fall in his Way, though more complex, without any further Aſſiſtance.

FINIS.

E R R A T A.

Note, when *b* is ſet to the Page, you muſt reckon from the Bottom.

Page	Line	Read	Page	Line	Read
6	3	$bqx^{m}y^{n-2}o^{2}\dot{y}^{2}$	253	1 *b.*	$\frac{\frac{x}{2a}A}{5}$
15	9	$x^{v} = \dot{z}$	260	6	Prop. II.)
	12	$\dot{z} = x^{v}y^{xv-1}\dot{y}$		11	Vertex
47	8	$-y^{2}x\dot{x}$		15	$\frac{\quad}{\overline{fc^{v}+gc^{w}}^{n}} \times \dot{x}$
125	6	$\frac{2R}{-\alpha}\sqrt{\alpha\beta z^{n}+\beta^{2}z^{2n}}$.	262	5	$\times \overline{a^{5}z-a^{6}}^{\frac{2}{3}}$
	7	$\frac{R}{\sqrt{-\alpha}}\sqrt{\alpha+\beta z^{n}}$	26	4 *b.*	$as^{\frac{\lambda}{s}}z^{\frac{\lambda}{s}}+$
166	1	Log: $\frac{\sqrt{rr-bb}+b\sqrt{\frac{rr-ss}{ss}}}{\sqrt{rr-bb}-b\sqrt{\frac{rr-ss}{ss}}}$:	264	10	$-\frac{c}{a}u^{r}$
178	2	(add) and Vel. of $w = .414 \times 2h$.	272	2 *b.*	$xx = cv$;
180	6	Log: $X -$ Log. x^{x}	278	2	Log: $\frac{dy}{bb}+$
227	6	$+m \times \overline{m-1} \times fx^{m-2}$	282	14	$= IR$;
	12	$m = 2,\ g =$	287	14	(when $s=0, y=a$)
	14	$\frac{-2ay^{3}\times\pi^{3}}{:2b\times\overline{-2ay}^{2}-2a\times\overline{2bx+ab}^{2}:\times y^{3}}$	288	7 *b.*	Fig. 130.
239	6 *b.*	dele (by Form the 16th)	291	3 *b.*	near ML,
241	7	$\underline{3-2d-dd}$		1 *b.*	$mo = Mr$,
244	6 *b.*	Fig. 59 (in the Margin)	292	8 *b.*	$=\frac{1}{2}v$.
246	17	$bc+cd-v$	294	2	Fig. 148.
248	14	$\frac{dyy}{2} =$		8 *b.*	dele 148 (Fig.)
250	1 *b.*	$BM \times \frac{DR}{2}$	295	4	Fig. 149
251	8	AbD		9	*the Curve is*
	9	Ab the		5 *b.*	that is $Mf -$
	16	$DR =$	296	12	*the Sphere*
253	1	$z = \frac{1}{m+1}y^{\frac{1}{m}+1}$	299	11	$+\overline{2mf-ma}\times$
				12	$\overline{\frac{ma}{b}-m}^{2}x^{2}$.
			300	3 *b.*	$\overline{SC-Sd}\times d$
			301	10	$= xs'$;

2 =

ERRATA.

Page	Line	Read
305	2	$2 = \frac{aay}{2}$ —
310	8 *b*.	or as $\frac{Cr}{CO^2}$.
313	1	Fig. 171.
315	7 *b*.	$= \frac{4rzz\dot{z}}{\sqrt{rr-zz}}$,
316	3 *b*.	$CQ^2 \times \dot{\overline{AQ}} =$
	ibid	$= x^2v + \frac{1}{3}v^3$.
317	13	$4\dot{\overline{SZ}} =$
319	9	$c \times 2Ef \times \dot{\overline{Ef}} \times Ef^2 =$
326	1	Fig. 201.
	13	*Cm* is
326	9 *b*.	*S.mCD* or
333	15	whence $\frac{\ddot{x}}{y\dot{y}^2}$.
334	11	*Br*, $\overset{\prime}{x}$;
392	5 *b*.	$\overline{1 - \frac{6}{5}v + 11.23v^2} \times$ In Page 265, 266, 267, 268, 269, read ε every where inſtead of *e*, except Line 5 *b* (p.266) read $\overline{m - 4p}.e$; and Line 4 *b* read $\overline{4p - 4\varepsilon - m - n} \times eA$.

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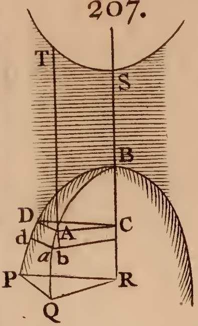

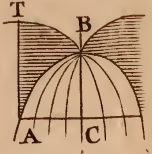

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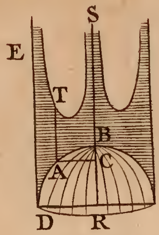

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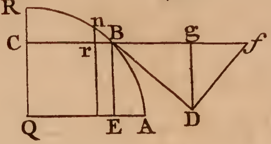

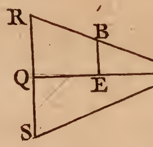

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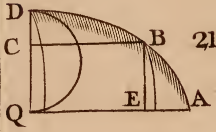

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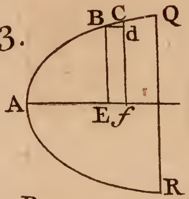

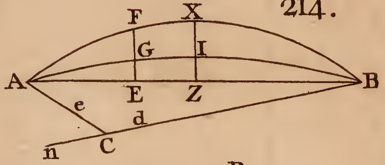

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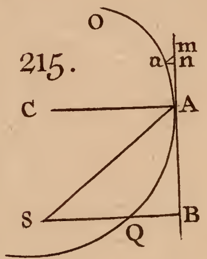

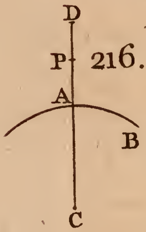

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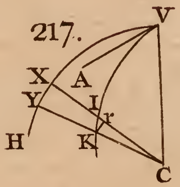

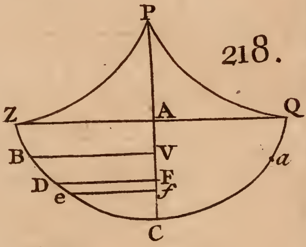

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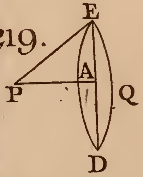

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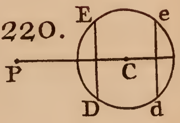

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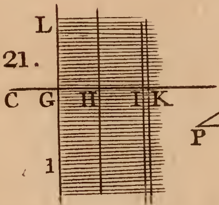

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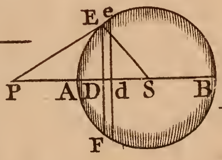

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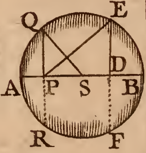

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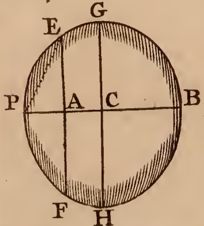

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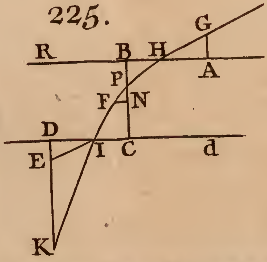

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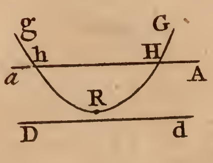

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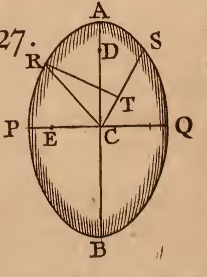
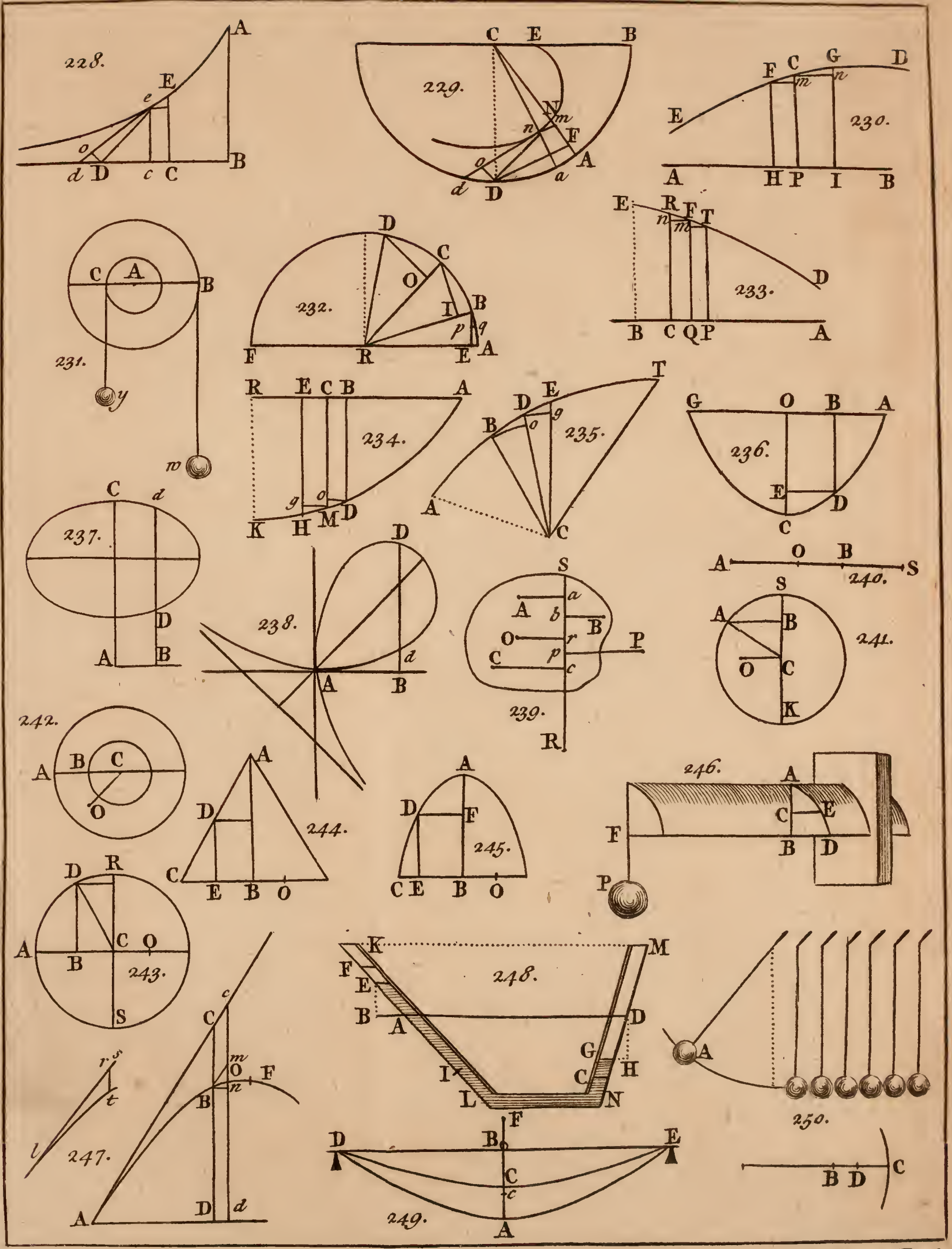


Pl. X. *the end.*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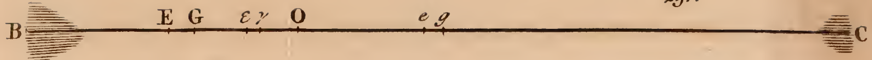
Pl. XI. *the End.*



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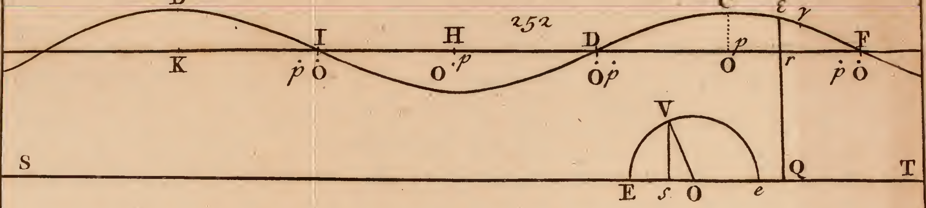

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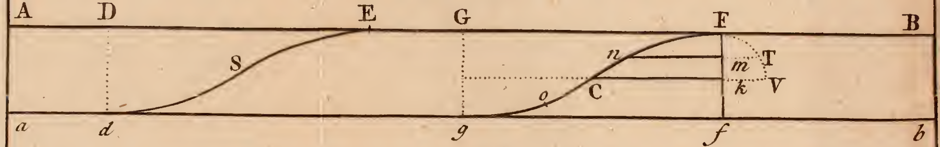

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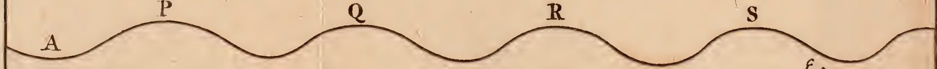

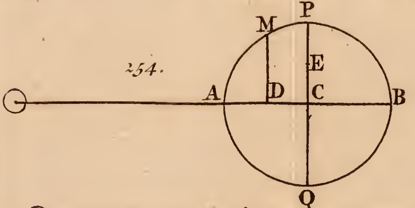

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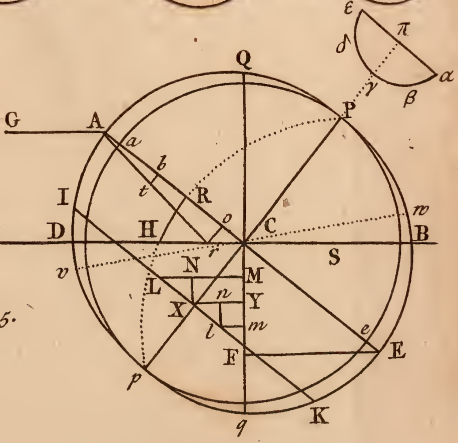

255.

Pl.XII. *the End.*

John Quincy Adams.

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